SVOLGIMENTI PROVA SCRITTA DI ANALISI 1 DEL 3/6/2026

1) $$a_n \sim \left[-\frac{1}{2n^2}\right] n^\alpha = -\frac{1}{2n^{2-\alpha}} < 0$$

$$a_n \longrightarrow \begin{cases} 0^- & \text{se } 2-\alpha>0, \text{ cioè } \alpha<2 \\ -1/2 & \text{se } \alpha=2 \\ -\infty & \text{se } \alpha>2 \end{cases}$$

La CN per la convergenza della serie è rispettata solo per $\alpha<2$.

$$\sum a_n \approx -\sum \frac{1}{2n^{2-\alpha}} \begin{cases} \text{convergente se } 2-\alpha>1 \text{ cioè } \alpha<1 \\ \text{divergente a } -\infty \text{ se } \alpha\geq 1 \end{cases}$$

2) $D=\mathbb{R}$ $\quad f\in C^0(\mathbb{R})$.

$f(x)\geq 0 \quad \forall x\in\mathbb{R} \quad f(x)=0 \iff x=0$

f pari, poichè $|x|$ è pari ed e^{-x^2} è pari.

Studiamo f solo per $x\geq 0$

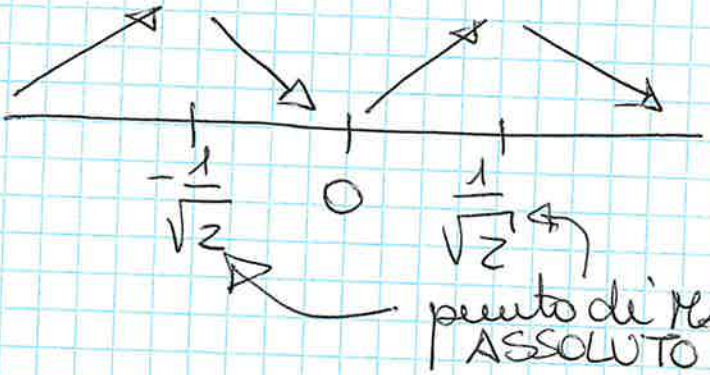
$$f(x)=xe^{-x^2} \qquad \lim_{x\to\pm\infty} f(x) = \lim_{x\to+\infty} \frac{x}{e^{x^2}} = 0^+$$

$y=0$ asintoto orizzontale a $\pm\infty$.

$$f'(x)=(1-2x^2)e^{-x^2} \geq 0 \iff \begin{cases} x>0 \\ 1-2x^2\geq 0 \end{cases}$$

$$\begin{cases} x>0 \\ -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \end{cases} \Rightarrow \quad 0 < x \le \frac{1}{\sqrt{2}}$$

Per simmetria



$$f\left(\pm\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} e^{-\frac{1}{2}}$$

Poiché ~~$\lim_{x\to\pm\infty} f(x) \ge 0$~~ $f(x) \ge 0 \Rightarrow x=0$ punto di MIN. ASSOLUTO

$\lim_{x\to 0^+} f'(x) = 1$ e, per simmetria,

$\lim_{x\to 0^-} f'(x) = -1$

$x=0$ punto di non derivabilità (angoloso).

Per $x>0$: $f'(x) = (-4x - 2x + 4x^3)e^{-x^2}$

$= \cancel{x(4x^2}$ $2x(2x^2-3)e^{-x^2} \ge 0$

$$\Leftrightarrow \begin{cases} x>0 \\ 2x^2-3 \ge 0 \end{cases} \Leftrightarrow \begin{cases} x>0 \\ x \le -\sqrt{\frac{3}{2}} \vee x \ge \sqrt{\frac{3}{2}} \end{cases}$$

Quindi, per $x>0$, f è concava in $\left(0, \sqrt{\frac{3}{2}}\right)$ e convessa in $\left(\sqrt{\frac{3}{2}}, +\infty\right)$.

Per simmetria, f convessa in $(-\infty, -\sqrt{\frac{3}{2}})$; concava in $(-\sqrt{\frac{3}{2}}, 0)$; concava in $(0, \sqrt{\frac{3}{2}})$, convessa in $(\sqrt{\frac{3}{2}}, +\infty)$.

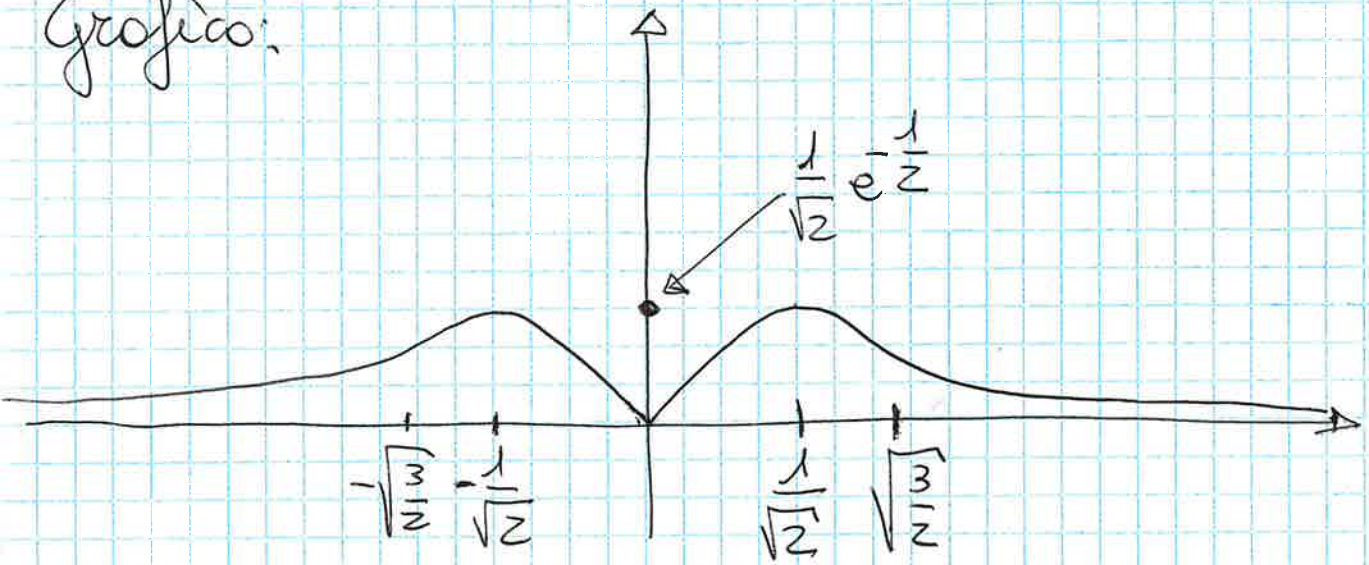
(3)

$x = \sqrt{\frac{3}{2}}$ punto di flesso ascendente

$x = -\sqrt{\frac{3}{2}}$ punto di flesso discendente.

$$f\left(\pm\sqrt{\frac{3}{2}}\right) = \sqrt{\frac{3}{2}}\, e^{-\frac{3}{2}}$$

Grafico:



3) $$\frac{z^3}{8} = \frac{(1-i)^3}{(1+i)^3(1-i)^3} = \frac{(1-i)^3}{2^3}$$

$$z^3 = \left[\sqrt{2}\left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)\right]^3 = 2\sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\,\mathrm{sen}\left(-\frac{\pi}{4}\right)\right]^3$$

$$= 2\sqrt{2}\left[\cos\left(-\frac{3}{4}\pi\right) + i\,\mathrm{sen}\left(-\frac{3}{4}\pi\right)\right]$$

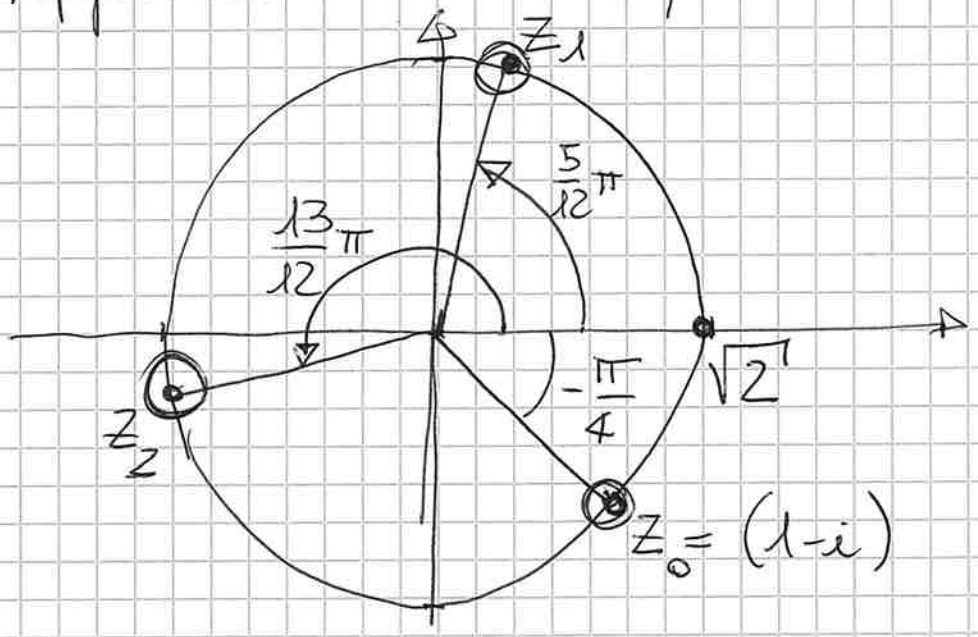
$$z_k = \sqrt{2}\left[\cos\left(\frac{-\frac{3}{4}\pi + 2k\pi}{3}\right) + i\,\text{sen}\left(\frac{-\frac{3}{4}\pi + 2k\pi}{3}\right)\right] \quad (4)$$

$$z_0 = \sqrt{2}\left[\cos\left(-\frac{\pi}{4}\right) + i\,\text{sen}\left(-\frac{\pi}{4}\right)\right] = \sqrt{2}\left[\frac{\sqrt{2}}{2} - i\,\frac{\sqrt{2}}{2}\right] = 1 - i$$

$$z_1 = \sqrt{2}\left[\cos\left(\frac{5}{12}\pi\right) + i\,\text{sen}\left(\frac{5}{12}\pi\right)\right]$$

$$z_2 = \sqrt{2}\left[\cos\left(\frac{13}{12}\pi\right) + i\,\text{sen}\left(\frac{13}{12}\pi\right)\right]$$

Approssimativamente, le soluzioni sono:



4) OMO. ASSOCIATA: $x^2 + 1 = 0$

$$x_{1,2} = \pm i$$

$$\Rightarrow y_0(x) = C_1 \cos x + C_2 \text{sen} x$$

SOL. PARTICOLARE della EQ. NON OMOGENEA CON IL METODO DI LAGRANGE

$$|W(\cos x, \operatorname{sen} x)| = \begin{vmatrix} \cos x & \operatorname{sen} x \\ -\operatorname{sen} x & \cos x \end{vmatrix} = 1$$ (5)

$$y_p(x) = C_1(x)\cos x + C_2(x)\operatorname{sen} x$$

$$C_1(x) = -\int \frac{\operatorname{sen} x}{\cos x}\,dx = \ln(|\cos x|)$$

$$C_2(x) = \int \frac{\cos x}{\cos x}\,dx = x$$

$$\Rightarrow y_{NO}(x) = C_1\cos x + C_2 \operatorname{sen} x + \ln(|\cos x|)\cos x + x\operatorname{sen} x.$$

5) $$f(x) = \frac{x^{\frac{3}{2}}\left[1-\left(1+\frac{x}{2}+o(x)\right)\left(1-\frac{x^2}{2}+o(x^2)\right)\right]}{\left[x\left(1+x+\frac{x^2}{2}+o(x^2)\right)-x-x^2\right]}$$

$$= \frac{x^{\frac{3}{2}}\left[1-\left(1+\frac{x}{2}+o(x)\right)\right]}{\left(x+x^2+\frac{x^3}{2}+o(x^2)-x-x^2\right)} \underset{x\to 0^+}{\sim} \frac{-x^{\frac{3}{2}}\frac{x}{2}}{\frac{x^3}{2}}$$

$$= -\frac{1}{x^{2-\frac{3}{2}}} = -\frac{1}{\sqrt{x}} = -\frac{1}{x^{\frac{1}{2}}}$$

Poiché $g(x) = -\frac{1}{\sqrt{x}}$ è integrabile in $(0,1]$, per il criterio del confronto asintotico, lo è anche $f(x)$.