SVOLGIMENTI PROVA SCRITTA di ANALISI 1 del 4/9/2023

1) $$(z+i) = x+i(y+1)$$

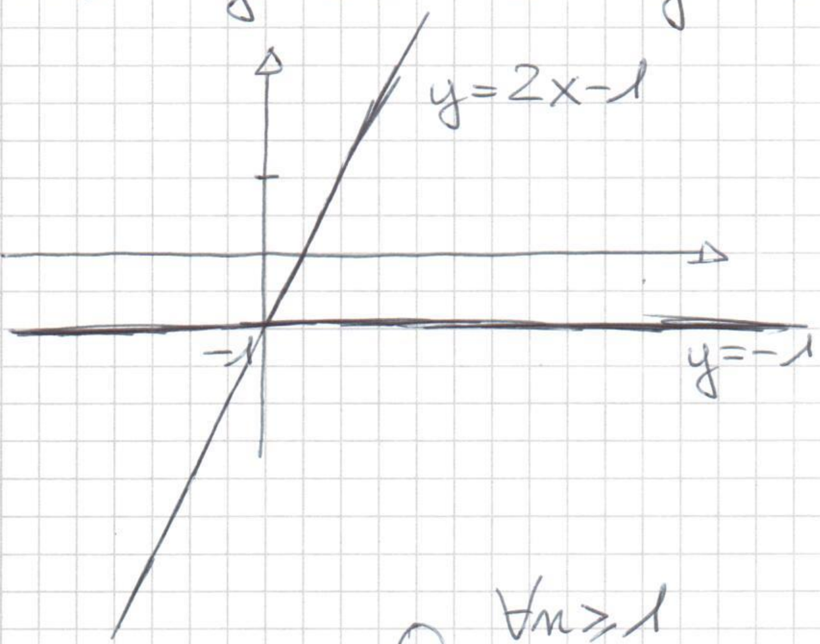
$$(z+i)^2 = x^2+2ix(y+1)-(y+1)^2$$

$$\operatorname{Im}\left[(z+i)^2\right] = 2x(y+1)$$

$$\left[\operatorname{Im}(z+i)\right]^2 = (y+1)^2$$

$$\Rightarrow (y+1)\left[2x-(y+1)\right] = 0$$

$$\Rightarrow \quad y=-1 \quad \vee \quad y=2x-1$$



2) $$\frac{n^2-1}{n^2+1} \geq 0 \quad \forall n \geq 1$$

$$\frac{n^2-1}{n^2+1} = 1 - \frac{2}{n^2+1}$$

$$\Rightarrow a_n = \left(1-\frac{2}{n^2+1}\right)^{n^4}$$

Criterio della radice: (2)

$$\sqrt[n]{a_n} = \left(1-\frac{2}{n^2+1}\right)^{\frac{n^4}{n}} = \left(1-\frac{2}{n^2+1}\right)^{n^3} =$$

$$\left[\left(1-\frac{2}{n^2+1}\right)^{-\left(\frac{n^2+1}{2}\right)}\right]^{-\frac{2n^3}{n^2+1}}$$

$$\Rightarrow \lim_{n\to\infty}\sqrt[n]{a_n} = \left[\lim\left(1-\frac{2}{n^2+1}\right)^{-\left(\frac{n^2+1}{2}\right)}\right]^{\lim_{n\to\infty}\left(\frac{-2n^3}{n^2+1}\right)}$$

$$= e^{-\infty} = 0 < 1. \Rightarrow \text{la serie CONVERGE}$$

3) $I_{def} = \{x \neq 0\}$.

$$\lim_{x\to 0^+} f(x) = e^{1/0^+} = e^{+\infty} = +\infty. \quad \text{AS. VERT. da SX} \quad x=0$$

$$\lim_{x\to 0^-} f(x) = e^{1/0^-} = e^{-\infty} = 0$$

$$\lim_{x\to\pm\infty} f(x) = e^{\frac{1}{\pm\infty}} \mp\infty = 1\mp\infty = \mp\infty$$

AS. OBLIQUO:

$$\lim_{x\to\pm\infty}\frac{f(x)}{x} = \lim_{x\to\pm\infty}\left[\frac{e^{\frac{1}{x}}}{x}-1\right] = -1 = m$$

$$\lim_{x\to\pm\infty}\left[f(x)-mx\right] = \lim_{x\to\pm\infty}\left[e^{1/x}-x+x\right] = \lim_{x\to\pm\infty} e^{1/x} = 1$$

$\Rightarrow$ AS. OBLIQUO a $\pm\infty$: $y = -x+1$.

In alternativa, avremmo potuto scrivere, per $x \to \pm\infty$,

(3)

$$f(x) = 1 + \frac{1}{x} + o\left(\frac{1}{x}\right) - x = -x + 1 + o(1)$$

$$\Rightarrow \quad f(x) \underset{x \to \pm\infty}{\sim} -x+1 \quad \Rightarrow \quad y = -x+1$$

AS. OBLIQUO a $\pm\infty$.

$$f'(x) = -\frac{1}{x^2} e^{1/x} - 1 < 0 \qquad \forall x \in I_{def}$$

f decresce in $(-\infty, 0)$ e in $(0, +\infty)$.

NO MAX-MIN, REL. o ASS.

Si osservi che $\lim_{x \to 0^-} f'(x) =$ $\left(\text{ponendo } t = -\frac{1}{x}\right)$

$$= \lim_{t \to +\infty} \frac{-t^2}{e^t} - 1 \quad \text{per la gerarchia degli infiniti}$$

$$= -1$$

$$f''(x) = e^{\frac{1}{x}} \left(\frac{1}{x^4} + \frac{2}{x^3}\right) = \frac{e^{\frac{1}{x}}}{x^4}(1+2x) > 0$$

$$\Leftrightarrow \quad x > -\frac{1}{2}$$

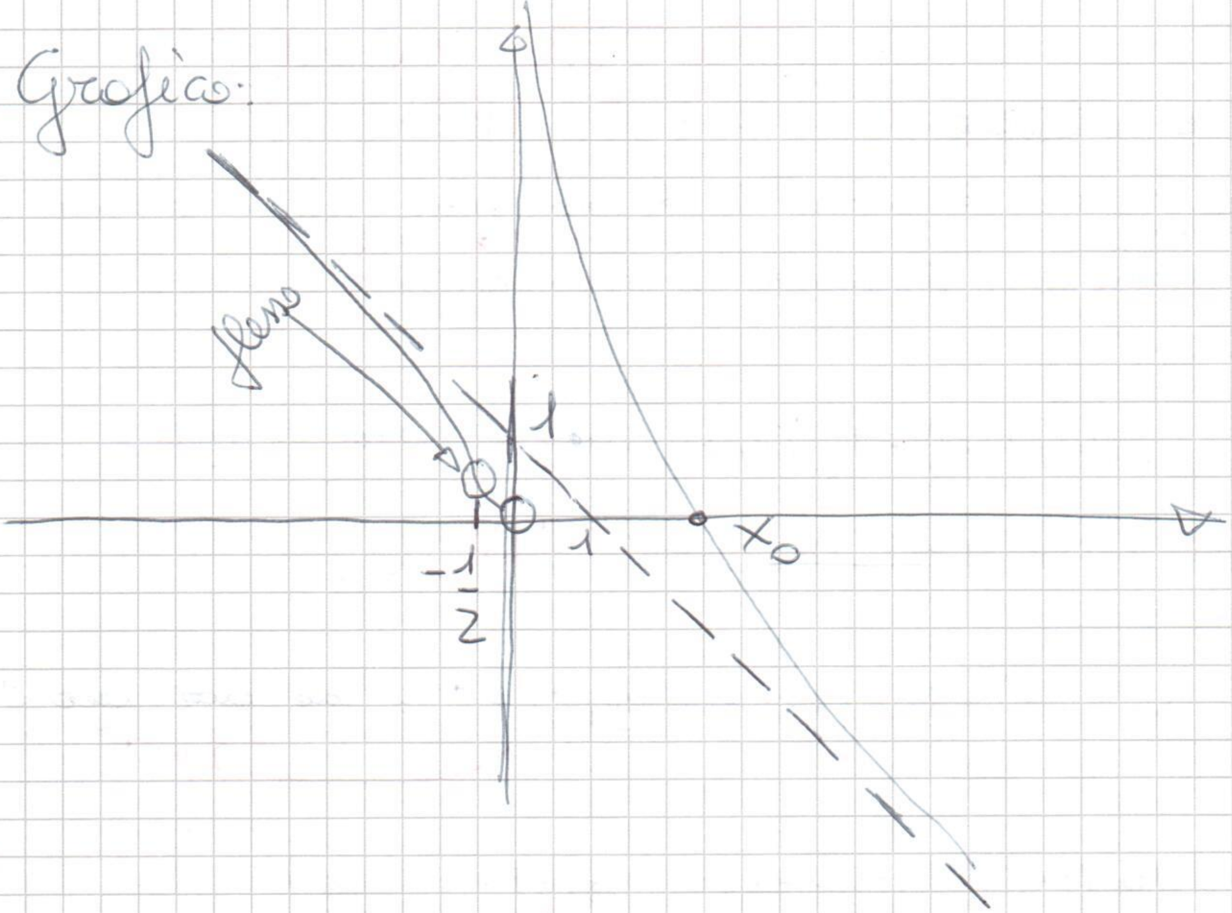
f concava in $\left(-\infty, -\frac{1}{2}\right)$; convessa in $\left(-\frac{1}{2}, 0\right)$ e in $(0, +\infty)$.

$x = -\frac{1}{2}$ punto di flesso ascendente

$$f\left(-\frac{1}{2}\right) = \frac{1}{2} + e^{-2}$$

Grafico:

④



Poiché f è strettamente monotona in $(0, +\infty)$,

$\lim_{x \to 0^+} f(x) = +\infty$; $\lim_{x \to +\infty} f(x) = -\infty$, allora

$\exists!\ x_0$ tale per cui $f(x_0) = 0$

Dal grafico si può osservare che $x_0 > 1$.

~~f~~ f positiva in $(-\infty, 0) \cup (0, x_0)$

negativa in $(x_0, +\infty)$.

4) Poniamo $t = x - 1$

$$\lim_{t \to 0} \frac{\sin t - \operatorname{arct} t}{[e^t - 1]^2 t} = \lim_{t \to 0} \frac{\cancel{t} - \frac{t^3}{6} - \cancel{t} + \frac{t^3}{3} + o(t^3)}{t^3}$$

$$\cancel{= \lim_{t \to 0} \frac{1}{6} = \frac{1}{6}}$$

$$= \lim_{t \to 0} \frac{\frac{t^3}{6} + o(t^3)}{t^3} = \frac{1}{6}$$

Il numeratore ha ordine di infinitesimo $\alpha = 3$ in $x = 1$.

5)

$$\int \frac{\operatorname{sen}^3 x}{\cos^2 x} = \int \left(\frac{1 - \cos^2 x}{\cos^2 x} \right) \operatorname{sen} x \, dx$$

$$= \int \frac{1}{\cos^2 x} \operatorname{sen} x \, dx - \int \operatorname{sen} x \, dx$$

$$= \frac{1}{\cos x} + \cos x$$

$$\Rightarrow \int_0^{\pi/4} f(x)\,dx = \left[\frac{1}{\cos x} + \cos x \right]_0^{\frac{\pi}{4}} = \sqrt{2} + \frac{\sqrt{2}}{2} - 2$$

$$= \frac{3}{2}\sqrt{2} - 2.$$

$$\int_0^{\pi/2} f(x)\,dx = \left[\frac{1}{\cos x} + \cos x \right]_0^{\left(\frac{\pi}{2}\right)^-} = (+\infty) - 2 = +\infty$$

Infine, $\int_{-\pi/4}^{\pi/4} f(x)\,dx = 0$ perché integrale di funzione dispari su intervallo simmetrico.