SVOLGIMENTI PROVA SCRITTA di ANALISI 1 DEL 9/9/2024

1)

$$\sin\left(\frac{1}{n^3}\right)=\frac{1}{n^3}-\frac{1}{6n^9}+o\left(\frac{1}{n^9}\right)$$

$$\ln\left(1-\frac{1}{n^3}\right)=-\frac{1}{n^3}-\frac{1}{2n^6}-\frac{1}{3n^9}+o\left(\frac{1}{n^9}\right)$$

$$\Rightarrow a_n=n^\alpha\left[\cancel{\frac{1}{2n^6}}+\cancel{\frac{1}{n^3}}-\frac{1}{6n^9}-\cancel{\frac{1}{n^3}}-\cancel{\frac{1}{2n^6}}-\frac{1}{3n^9}+o\left(\frac{1}{n^9}\right)\right]$$

$$\sim n^\alpha\left(-\frac{1}{3}-\frac{1}{6}\right)\frac{1}{n^9}=-\frac{1}{2n^{9-\alpha}}$$

Serie definitivamente negativa (quindi di segno costante). Applico il criterio del confronto asintotico

lo serie $\begin{cases}\text{converge se } 9-\alpha>1 \text{ cioè } \alpha<8\\ \text{diverge negativamente se } \alpha\geq 8\end{cases}$

2) DOMINIO: $D=\{x\neq\pm\sqrt{3}\}=(-\infty,-\sqrt{3})\cup(-\sqrt{3},\sqrt{3})\cup(\sqrt{3},+\infty)$.

f è DISPARI:

$$f(-x)=\frac{(-x)^3}{3-(-x)^2}=-\frac{x^3}{3-x^2}=-f(x).$$

$$f(x) = -\left[\frac{x^3}{(x-\sqrt{3})(x+\sqrt{3})}\right]$$

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$$\lim_{x\to\sqrt{3}^\pm} f(x) = \frac{-(\sqrt{3})^3}{0^\pm \cdot 2\sqrt{3}} = \mp\infty.$$

Per disparità, $\lim_{x\to-\sqrt{3}^\pm} = \pm\infty$.

$x = \pm\sqrt{3}$ sono ASINTOTI VERTICALI.

$$f(x) = -\left[\frac{x^3}{x^2-3}\right] = -\left[\frac{x^3-3x+3x}{x^2-3}\right]$$

$$= -x - \frac{3x}{x^2-3} = -x + o(1)$$

$\Rightarrow \lim_{x\to\pm\infty} f(x) = \mp\infty$. Inoltre $y=-x$ è ASINTOTO OBLIQUO ~~ORIZZONTALE~~ a $\pm\infty$.

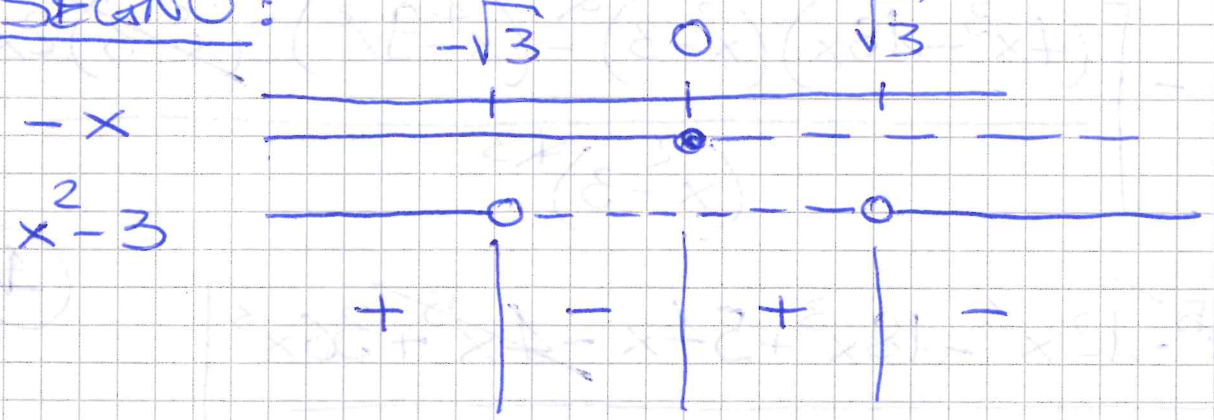
INTERSEZIONI CON ASSI:

$f(0)=0$

$f(x)=0 \iff x=0$

Unica intersezione nell'origine.

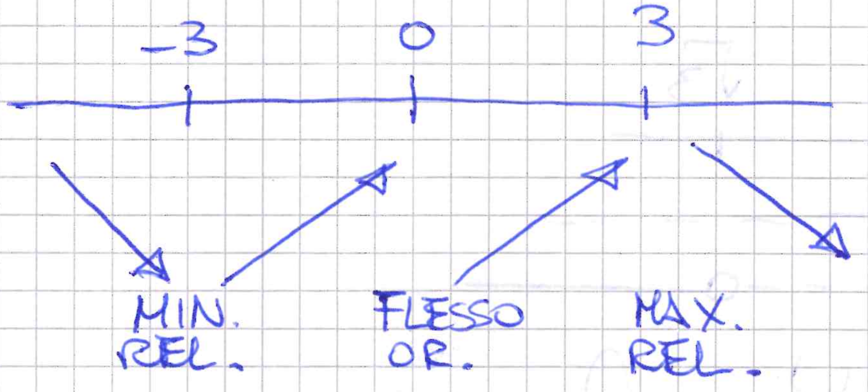
SEGNO:



$$f'(x) = -\left[\frac{3x^2(x^2-3) - x^3\cdot 2x}{(x^2-3)^2}\right] = -\left[\frac{x^4-9x^2}{(x^2-3)^2}\right]$$

$$= -x^2\left[\frac{x^2-9}{(x^2-3)^2}\right] = 0 \iff x=0\ ;\ x=\pm 3.$$

$$f'(x) > 0 \iff x^2-9<0 \iff x\in(-3,3)$$



$$f(\pm 3) = \frac{\pm 27}{-6} = \mp\frac{9}{2}$$

Poichè $\lim\limits_{x\to\pm\infty} f(x) = \mp\infty \Rightarrow \nexists$ MAX. o MIN. ASS.

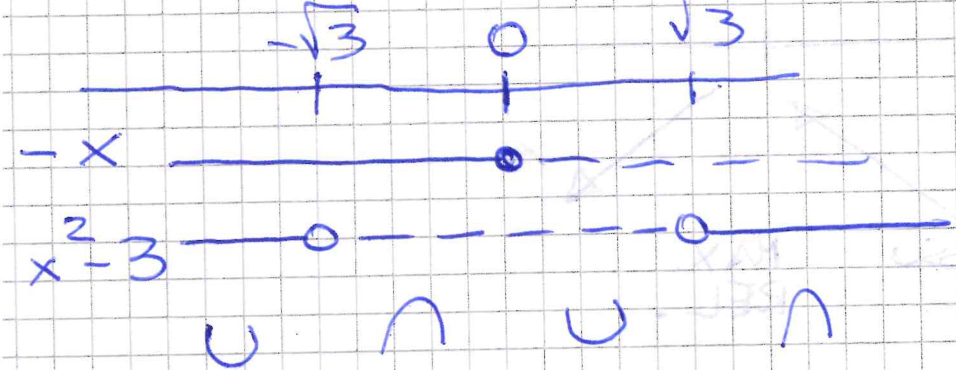
$$f''(x) = -\left[\frac{(4x^3-18x)(x^2-3)^{\cancel{2}} - (x^4-9x^2)\,2\cancel{(x^2-3)}\,2x}{(x^2-3)^{\cancel{4}\,3}}\right]$$

$$= -\left[\frac{\cancel{4x^5} - 12x^3 - 18x^3 + 54x - \cancel{4x^5} + 36x^3}{(x^2-3)^3}\right]$$

$$= -\left[\frac{6x^3+54x}{(x^2-3)^3}\right] = -6x\left[\frac{x^2+9}{(x^2-3)^3}\right]$$

$f''(x) = 0 \iff x = 0$ (flesso a tangente orizzontale, già individuato)

SEGNO di f'':



f è CONVESSA in $(-\infty,-\sqrt{3})$ e in ~~$(\sqrt{3},+\infty)$~~ $(0,\sqrt{3})$; è CONCAVA in $(-\sqrt{3},0)$ e $(\sqrt{3},+\infty)$.

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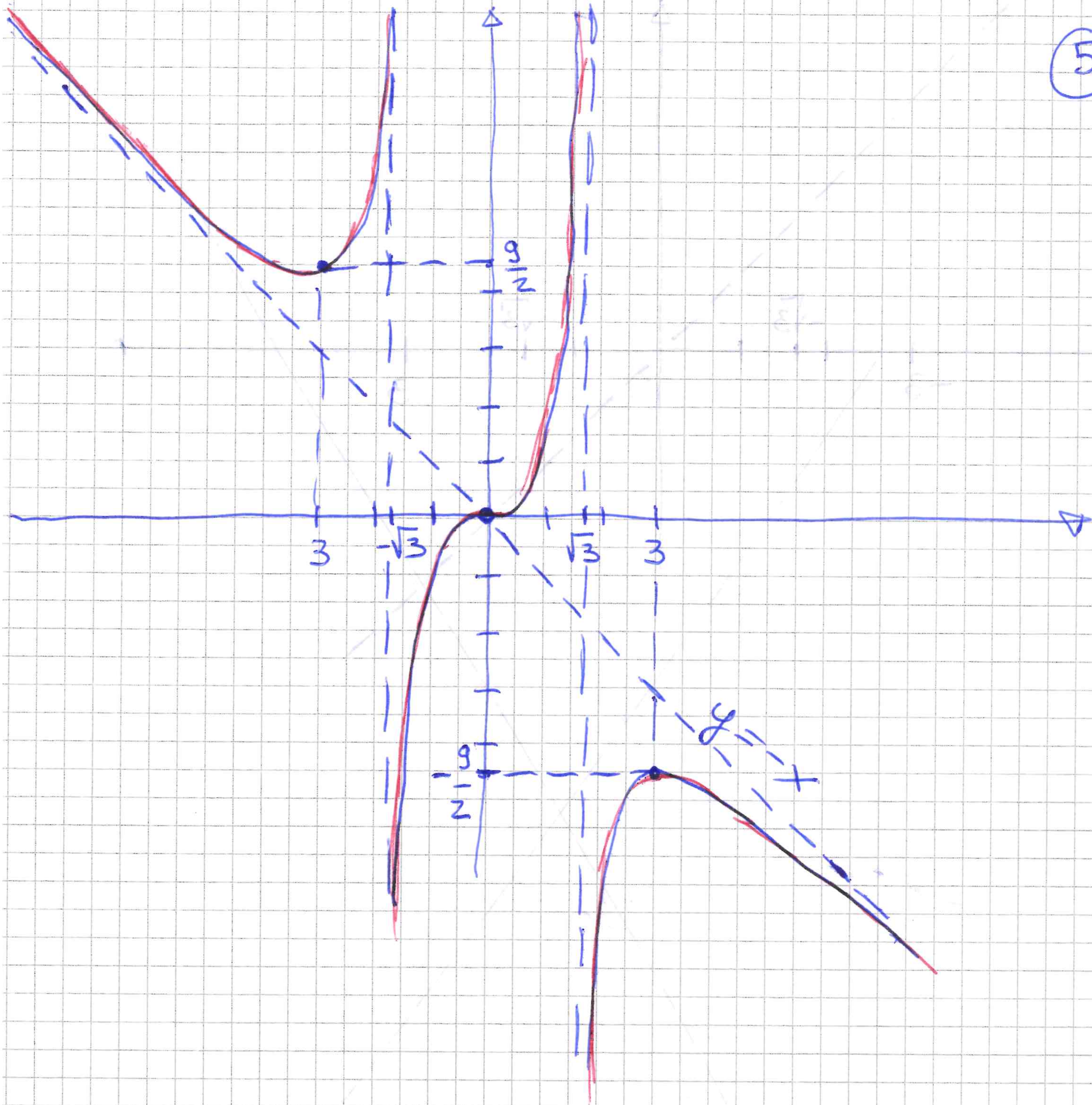

3)

$$\frac{1-i}{1+i} = \frac{(1-i)^2}{(1+i)(1-i)} = \frac{\cancel{1}-\cancel{1}-2i}{1+1} = -i$$

$$= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right)$$

$$\sqrt[3]{-i} = \cos\left(\frac{-\frac{\pi}{2}+2k\pi}{3}\right) + i\sin\left(\frac{-\frac{\pi}{2}+2k\pi}{3}\right)$$

$k = 0, 1, 2$

$k=0$:

$$z_0 = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$$
$$= \frac{\sqrt{3}}{2} - i\,\frac{1}{2}$$

$$z_2 = \cos\left(\frac{7}{6}\pi\right) + i\sin\left(\frac{7}{6}\pi\right) = \frac{-\sqrt{3}}{2} - i\,\frac{1}{2}$$

$$z_1 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i$$

4) omogenea associata:

$$\alpha^2 + \alpha = 0 \qquad \Rightarrow \alpha_1 = 0\,;\ \alpha_2 = -1$$

$$y_0(x) = C_1 + C_2 e^{-x}$$

$$y_p(x) = A\cos x + B\sin x$$

$$y_p'(x) = -A\sin x + B\cos x$$

$$y_p''(x) = -A\cos x - B\sin x$$

$$-A\cos x - B\sin x - A\sin x + B\cos x = 2\cos x + \sin x$$

$$\begin{cases} -A + B = 2 \\ -B - A = 1 \end{cases} \qquad \begin{cases} A = -\dfrac{3}{2} \\ B = \dfrac{1}{2} \end{cases}$$

$$\Rightarrow y_{NO}(x) = C_1 + C_2 e^{-x} - \frac{3}{2}\cos x + \frac{1}{2}\sin x$$

$$y'(x) = -C_2 e^{-x} + \frac{3}{2}\sin x + \frac{1}{2}\cos x$$

$$\begin{cases} y(0) = C_1 + C_2 - \dfrac{3}{2} = 0 \\ y'(0) = -C_2 + \dfrac{1}{2} = 0 \end{cases} \qquad \begin{cases} C_2 = \dfrac{1}{2} \\ C_1 = \dfrac{3}{2} - C_2 = 1 \end{cases}$$

$$\Rightarrow y(x) = 1 + \frac{1}{2}e^{-x} - \frac{3}{2}\cos x + \frac{1}{2}\sin x.$$

<u>In alternativa:</u>

$$z = y'$$

$$\Rightarrow \begin{cases} z' + z = 2\cos x + \sin x \\ z(0) = 0 \end{cases}$$

$$\Rightarrow z(x) = e^{-x}\left[\int_0^x e^t (2\cos t + \sin t)\,dt\right]$$

$$\int e^t \cos t\, dt = e^t \sin t - \int e^t \sin t\, dt$$

$$= e^t \sin t - \left[-e^t \cos t + \int e^t \cos t\, dt \right]$$

$$= e^t \sin t + e^t \cos t - \int e^t \cos t\, dt$$

$$\Rightarrow \underline{\int e^t \cos t\, dt = \frac{1}{2} e^t (\sin t + \cos t)}$$

$$\int e^t \sin t\, dt = -e^t \cos t + \int e^t \cos t\, dt$$

$$= -e^t \cos t + e^t \sin t - \int e^t \sin t\, dt$$

$$\Rightarrow \int e^t \sin t\, dt = \frac{1}{2} e^t (\sin t - \cos t).$$

$$\Rightarrow z(x) = e^{-x} \left[e^t \left(\sin t + \cos t + \frac{1}{2} \sin t - \frac{1}{2} \cos t \right) \right]_0^x$$

$$= e^{-x} \left[e^t \left(\frac{3}{2} \sin t + \frac{1}{2} \cos t \right) \right]_0^x$$

$$= e^{-x} \left[e^x \left(\frac{3}{2} \sin x + \frac{1}{2} \cos x \right) - \frac{1}{2} \right]$$

$$= \frac{3}{2} \sin x + \frac{1}{2} \cos x - \frac{1}{2} e^{-x}.$$

$$y(x) = \int_0^x z(t)\,dt = \int_0^x \left[\frac{3}{2}\sin t + \frac{1}{2}\cos t - \frac{1}{2}e^{-t}\right]dt$$

$$= \left[-\frac{3}{2}\cos t + \frac{1}{2}\sin t + \frac{1}{2}e^{-t}\right]_0^x$$

$$= -\frac{3}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{2}e^{-x} + \frac{3}{2} - \frac{1}{2}$$

$$= -\frac{3}{2}\cos x + \frac{1}{2}\sin x + \frac{1}{2}e^{-x} + 1.$$

5) $f(x) \underset{x \to 0}{\sim} \frac{1}{\sqrt{x}} = \frac{1}{x^{1/2}}$ pertanto è integrabile in $[0,1]$.

$$\int_0^1 \left[\frac{2^{\sqrt{x}} + \sqrt{x}}{\sqrt{x}}\right]dx = \int_0^1 \left[\frac{2^{\sqrt{x}}}{\sqrt{x}} + 1\right]dx$$

$$\left[\frac{1}{\ln 2}\, 2 \cdot 2^{\sqrt{x}} + x\right]_0^1 = \frac{4}{\ln 2} + 1 - \frac{2}{\ln 2}$$

$$= \frac{2}{\ln 2} + 1 = 2\log_2 e + 1.$$