SVOLGIMENTI PROVA SCRITTA

di ANALISI 1 del 10/6/2021 1

1) $I_{def} = \{x \geq 0\}$

$f(x) \geq 0 \quad \forall x \in I_{def}$

$f(\cancel{x}) = 0 \iff x = 0$

$$f'(x) = e^{2\sqrt{x}}\left[1 + x\frac{1}{\sqrt{x}}\right] = \underbrace{e^{2\sqrt{x}}(1+\sqrt{x})}_{\forall x > 0}$$

NON DEFINITA in $x=0$

$$f'_+(0) = \lim_{x \to 0^+} f'(x) = 1$$

f derivabile (solo da DX) anche in $x=0$.

$f'(x) > 0 \quad \forall x \in I_{def} \Rightarrow f$ sempre crescente

$$\lim_{x \to +\infty} f(x) = +\infty$$

Andamento superlineare:

$$\lim_{x \to +\infty} \frac{f(x)}{x} = \lim_{x \to +\infty} e^{2\sqrt{x}} = +\infty.$$

NO AS. OBLIQUO.

(2)

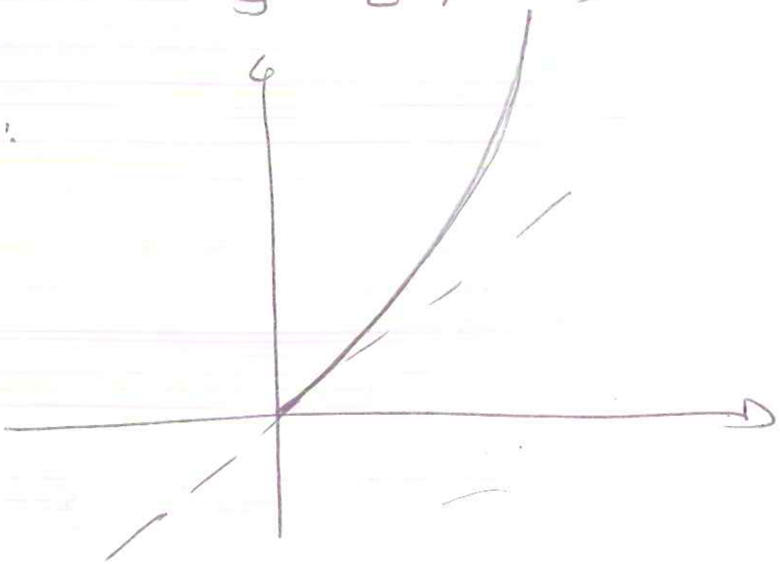
$$f''(x) = e^{2\sqrt{x}}\left[\frac{1}{2\sqrt{x}} + (1+\sqrt{x})\frac{1}{\sqrt{x}}\right]$$

$$= e^{2\sqrt{x}}\left[\frac{3}{2\sqrt{x}} + 1\right] > 0 \quad \forall x > 0$$

f sempre convessa.

Immagine: $R_f = [0, +\infty)$

Grafico:



2)

$$\lim_{x\to 0} \frac{\ln\left(\cos x + \frac{x^2}{2} - \frac{x^4}{24}\right)}{\sin(x^2) - \mathrm{arctg}(x^2)}$$

$$= \lim_{x\to 0} \frac{\ln\left[1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + o(x^6) + \frac{x^2}{2} - \frac{x^4}{24}\right]}{\left(x^2 - \frac{x^6}{3!}\right) - \left(x^2 - \frac{x^6}{3}\right) + o(x^6)}$$

$$= \lim_{x\to 0} \frac{\ln\left(1-\frac{x^6}{6!}\right)}{\left(-\frac{1}{6}+\frac{1}{3}\right)x^6} = \lim_{x\to 0} \frac{-\frac{x^6}{720}}{\frac{1}{6}x^6}$$ 3

$$= -\frac{1}{120}.$$

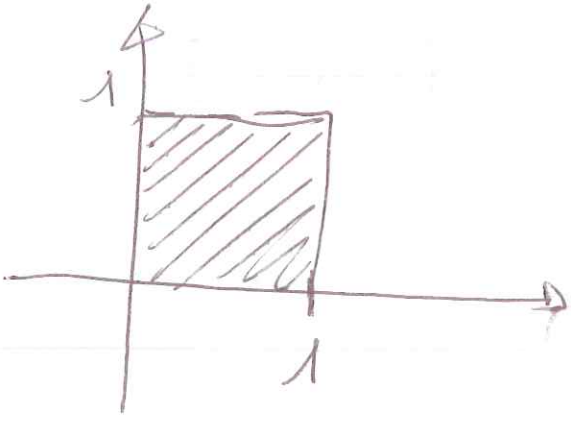
3) $\bar{z}(z+1) = x^2+y^2+x-iy$

$\operatorname{Im}\left(\bar{z}(z+1)\right) = -y$

$i(z-i) = ix-y+1$

$\operatorname{Im}\left(i(z-i)\right) = x$

$$\Rightarrow \begin{cases} -1 \le -y \le 0 \\ 0 \le x \le 1 \end{cases} \Rightarrow \begin{cases} 0 \le y \le 1 \\ 0 \le x \le 1 \end{cases}$$



4) $$a_n \sim \frac{n^n}{\frac{(2n)^{2n}}{e^{2n}}\sqrt{2\pi \cdot 2n}}$$ (4)

$$= \frac{e^{2n} \cdot n^n}{4^n \cdot n^{2n}\sqrt{4\pi n}} = \frac{\left(\frac{e^2}{4}\right)^n}{n^n} \cdot \frac{1}{\sqrt{4\pi n}}$$

Ma $\frac{a^n}{n^n} \longrightarrow 0 \quad \forall a > 0$

$$\Rightarrow \lim_{n\to\infty} a_n = 0$$

Studiamo la convergenza assoluta della serie:

$$\sum |(-1)^n a_n| = \sum a_n \approx \sum \underbrace{\frac{\left(\frac{e^2}{4}\right)^n}{n^n\sqrt{4\pi n}}}_{b_n}$$

Criterio della radice

$$\sqrt[n]{b_n} = \frac{e^2}{4n\sqrt[n]{\sqrt{4\pi n}}} \longrightarrow 0 < 1$$

$\Rightarrow$ la serie converge assolutamente e quindi semplicemente

⑤

~~Si noti che la convergono a 0~~

Altrimenti, studiando direttamente

$$\sum a_n = \sum \frac{n^n}{(2n)!}$$

Criterio del rapporto:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{n^n}$$

$$= \frac{(n+1)^n}{n^n} \cdot \frac{(n+1)\,(2n)!}{(2n+2)!}$$

$$= \left(1+\frac{1}{n}\right)^n \frac{(n+1)\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \left(1+\frac{1}{n}\right)^n \frac{1}{2(2n+1)} \longrightarrow \frac{e}{2\cdot(+\infty)} = 0 < 1$$

Si noti che si sarebbe potuto ricavare $a_n \xrightarrow[n\to\infty]{} 0$ dal fatto che $\sum a_n$ converge.

Infatti, se $\sum a_n$ converge $\Rightarrow a_n \to 0$.

5) $$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{(1-\cos^2 x)\cos x}\, dx$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\sin^2 x \cdot \cos x}\, dx = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \underbrace{|\sin x| \sqrt{\cos x}}_{\text{PARI}}\, dx$$

$$= 2 \int_0^{\frac{\pi}{2}} \sin x \cdot \sqrt{\cos x}\, dx$$

$$= -\frac{4}{3} (\cos x)^{\frac{3}{2}} \Big|_0^{\frac{\pi}{2}} = \frac{4}{3}.$$

N.B.: si sarebbe potuto effettuare anche un cambio di variabile: (6b)

$$\cos x = t \implies dt = -\sin x \, dx$$

In tal caso, però, sappiamo che

$$\sin x = \sqrt{1-\cos^2 x} \quad \text{se } x \in \left[0, \tfrac{\pi}{2}\right]$$
$$= -\sqrt{1-\cos^2 x} \quad \text{se } x \in \left[-\tfrac{\pi}{2}, 0\right]$$

$$\implies dx = \begin{cases} \dfrac{-dt}{\sqrt{1-t^2}} & \text{se } x \in \left[0, \frac{\pi}{2}\right] \\[2ex] \dfrac{dt}{\sqrt{1-t^2}} & \text{se } x \in \left[-\frac{\pi}{2}, 0\right] \end{cases}$$

$$\implies \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{\cos x - \cos^3 x}\, dx = \int_{-\frac{\pi}{2}}^{0} \sqrt{}\, dx + \int_{0}^{\frac{\pi}{2}} \sqrt{}\, dx$$

$$= \int_0^1 \frac{\sqrt{t-t^3}}{\sqrt{1-t^2}}\, dt + \int_1^0 \frac{-\sqrt{t-t^3}}{\sqrt{1-t^2}}\, dt$$

$$= 2\int_0^1 \frac{\sqrt{t\,\cancel{(1-t^2)}}}{\cancel{\sqrt{1-t^2}}}\, dt = 2\left[\frac{2}{3} t^{\frac{3}{2}}\right]_0^1 = \frac{4}{3}.$$

5 bis)

Eq.ne lineare del 1° ordine a coefficienti continui.

$$-3x^2 \in C^\infty(\mathbb{R})$$
$$x^5+x^2 \in C^\infty(\mathbb{R})$$

$\Rightarrow \exists!$ sol. (di tipo globale) su tutto $\mathbb{R}$.

$$y(x) = e^{\int_0^x 3t^2\,dt}\left[\int_0^x e^{-\int_0^t 3s^2\,ds}(t^5+t^2)\,dt + \frac{1}{3}\right]$$

$$= e^{t^3\big|_0^x}\left[\frac{1}{3} + \int_0^x e^{-s^3\big|_0^t}(t^5+t^2)\,dt\right]$$

$$= e^{x^3}\left[\frac{1}{3} + \underbrace{\int_0^x e^{-t^3}(t^3+1)\,t^2\,dt}\right]$$

$$t^3 = u \Rightarrow du = 3t^2\,dt$$
$$u(0)=0 \ ; \ u(x)=x^3$$

$$= e^{x^3}\left[\frac{1}{3} + \int_0^{x^3} \frac{e^{-u}(u+1)}{3}\,du\right] =$$

$$\frac{1}{3}e^{x^3}\left[1 - e^{-u}(u+1)\Big|_0^{x^3} + \int_0^{x^3} e^{-u}\,du\right]$$

$$= \frac{1}{3}e^{x^3}\left[1 - e^{-x^3}(x^3+1) + 1 - e^{-u}\Big|_0^{x^3}\right]$$

$$= \frac{1}{3}e^{x^3}\left[1 - e^{-x^3}(x^3+1) + 1 - e^{-x^3} + 1\right]$$

$$= -\frac{1}{3}(x^3+2) + e^{x^3}$$