SVOLGIMENTI PROVA SCRITTA di ANALISI 2 del 10/7/2025

1) ~~Il denominato~~ Ovviamente, $x \geq 0$. In tal caso, il denominatore è sempre strettamente positivo.

$\Rightarrow D_c = [0, +\infty)$; $\varphi_n(x) \geq 0 \quad \forall x \in D_c$.

$$\varphi_n(0) = 0 \xrightarrow[n\to\infty]{} 0$$

$$\forall x > 0 \quad \varphi_n(x) \underset{n\to\infty}{\sim} \frac{n^2\sqrt{x}}{n^2} \xrightarrow[n\to+\infty]{} \sqrt{x}$$

$$\Rightarrow D_{cp} = D_c = [0, +\infty) \qquad \varphi(x) = \sqrt{x}$$

$$|\varphi_n(x) - \varphi(x)| = \left| \frac{n^2\sqrt{x}}{\sqrt{x}+n^2} - \sqrt{x} \right| = \left| \frac{\cancel{n^2\sqrt{x}} - x - \cancel{n^2\sqrt{x}}}{\sqrt{x}+n^2} \right|$$

$$= \frac{x}{\sqrt{x}+n^2}$$

$$\sup_{[0,+\infty)} |\varphi_n - \varphi| = \sup_{[0,+\infty)} \frac{x}{\sqrt{x}+n^2} \geq \varphi_n(n^2) = \frac{n^2}{n^2+n} \xrightarrow[n\to+\infty]{} 1$$

NO CONV. UNIF. in $[0, +\infty)$.

In $[0, \alpha]$:

$$|\varphi_n(x) - \varphi(x)| = \frac{x}{\sqrt{x}+n^2} \leq \frac{\alpha}{n^2} \xrightarrow[n\to+\infty]{} 0$$

$\Rightarrow$ CONV. UNIF. in ogni $[0, \alpha]$.

In alternativa:

$$|\varphi_n(x) - \varphi(x)|' = \left(\frac{x}{\sqrt{x}+n^2} \right)' = \frac{\sqrt{x}+n^2 - \frac{x}{2\sqrt{x}}}{(\sqrt{x}+n^2)^2}$$

$$= \frac{\frac{1}{2}\sqrt{x} + n^2}{(\sqrt{x} + n^2)^2} > 0 \qquad \forall x \in [0, +\infty)$$

$$\Rightarrow |\varphi_n(x) - \varphi(x)| \text{ crescente in } [0, +\infty)$$

$$\Rightarrow \sup_{[0,+\infty)} |\varphi_n(x) - \varphi(x)| = \lim_{x \to +\infty} \frac{x}{\sqrt{x} + n^2} = +\infty.$$

In $[0, \alpha]$: $\quad \sup_{[0,\alpha]} |\varphi_n(x) - \varphi(x)| = \left. \frac{x}{\sqrt{x} + n^2} \right|_{x=\alpha} = \cancel{\alpha}$

$$= \frac{\alpha}{\sqrt{\alpha} + n^2} \xrightarrow[n \to +\infty]{} 0.$$

In particolare, si ha CONV. UNIF. in $[0, 1]$

$$\Rightarrow \lim_{n \to +\infty} \int_0^1 \varphi_n(x)\,dx = \int_0^1 \lim_{n \to +\infty} \varphi(x)\,dx$$

$$= \int_0^1 \sqrt{x}\,dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}.$$

Si osservi inoltre che $\displaystyle\int_0^{+\infty} \varphi_n(x)\,dx = +\infty \quad \forall n \in \mathbb{N}$,

in quanto $\displaystyle\lim_{x \to +\infty} \varphi_n(x) = \lim_{n \to +\infty} \frac{n^2\sqrt{x}}{\sqrt{x} + n^2} = n^2$.

Pertanto, la successione $\left\{ \displaystyle\int_0^{+\infty} \varphi_n(x)\,dx \right\}$ non è neanche definita...

2) $$|f(x,y)| = \left| \frac{\cancel{1} + \frac{(x^2+y^2)^2}{2} + o\left((x^2+y^2)^2\right) + x^3 - \cancel{1}}{x^2+y^2} \right|$$ 3

$$\approx \frac{\left|\frac{\rho^4}{2} + \rho^3 \cos^3\theta\right|}{\rho^2} \le \frac{\rho^2}{2} + \rho \xrightarrow[\rho \to 0]{} 0$$

f continua in $(0,0)$.

$$f(0,y) = \frac{\sqrt{1+y^4}-1}{y^2} = \frac{\cancel{1} + \frac{y^4}{2} + o(y^4) - \cancel{1}}{y^2}$$

$$\boxed{\frac{\partial f}{\partial y}(0,0)} = \lim_{k \to 0} \frac{\frac{k^4}{2} + o(k^4)}{k^3} = \boxed{0.}$$

$$f(x,0) = \frac{\sqrt{1+x^4}+x^3-1}{x^2} = \frac{\cancel{1} + \frac{x^4}{2} + x^3 - \cancel{1} + o(x^4)}{x^2}$$

$$\boxed{\frac{\partial f}{\partial x}(0,0)} = \lim_{h \to 0} \frac{h^3 + o(h^3)}{h^3} = \boxed{1.}$$

$$\frac{df}{d\vec{v}}(0,0) = \lim_{t \to 0} \frac{\sqrt{1+t^4} + t^3\alpha^3 - 1}{t^3} = \lim_{t \to 0} \frac{\cancel{1} + \frac{t^4}{2} + t^3\alpha^3 - \cancel{1}}{t^3}$$

$$= \lim_{t \to 0} \frac{t^3\alpha^3}{t^3} = \alpha^3$$ $\exists$ DER. DIR. $\forall \vec{v}$ $\cancel{\forall(\alpha,\beta)}$

No formula del gradiente $\Rightarrow f$ NON differenziabile in $(0,0)$.

3) $f(x,y) = (y-1)^2 - 4x^2$ (4)

è un paraboloide iperbolico centrato in $(0,1)$, che, come noto, è punto di sella.

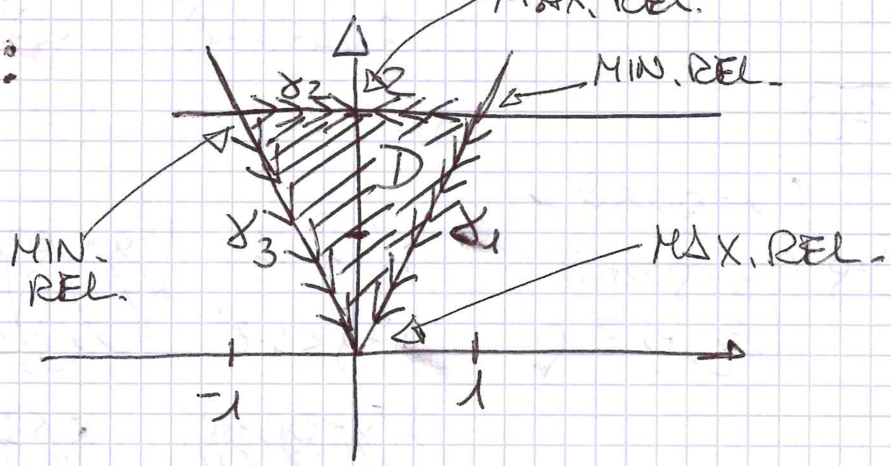
Infatti:

$$\begin{cases} f_x = -8x = 0 \\ f_y = 2y-2 = 0 \end{cases} \Rightarrow \begin{cases} x=0 \\ y=1 \end{cases}$$

~~$f_{xx} = -8$~~

$$H_f = \begin{pmatrix} -8 & 0 \\ 0 & 2 \end{pmatrix}$$ matrice indefinita.

∂D:



$\partial D = \gamma_1 \cup \gamma_2 \cup \gamma_3$

$\gamma_1: \begin{cases} x \in [0,1] \\ y = 2x \end{cases}$; $\gamma_2: \begin{cases} y=2 \\ x \in [-1,1] \end{cases}$; $\gamma_3: \begin{cases} x \in [-1,0] \\ y = -2x \end{cases}$

$f|_{\gamma_1} = \cancel{4x^2} - 4x - \cancel{4x^2} + 1$ funzione decrescente

$f|_{\gamma_2} = 1 - 4x^2$ parabola rivolta verso il basso
$(0,2)$ punto di MAX. REL.
$f(0,2) = 1$

$f|_{\gamma_3} = \cancel{4x^2} + 4x - \cancel{4x^2} + 1$ funzione crescente

Gli andamenti sono coerenti con la simmetria della f rispetto all'asse y:

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$$f(-x,y)=f(x,y).$$

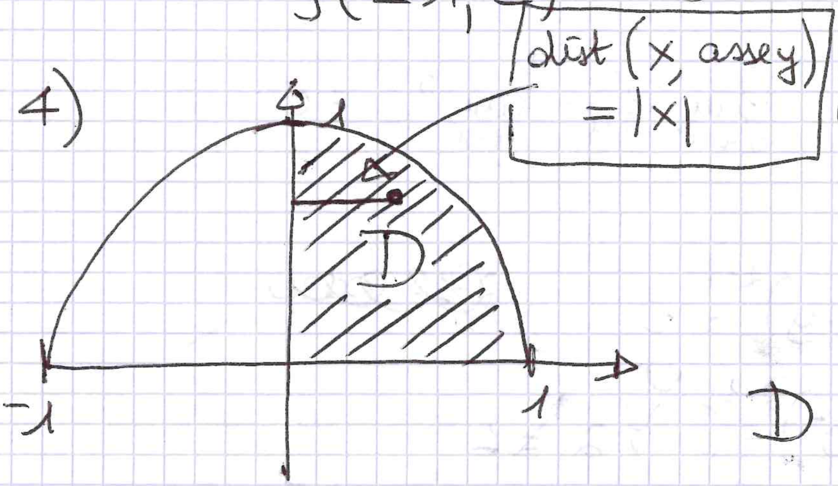
MAX. ASS. in $(0,0)$ e $(0,2)$:

$$f(0,0)=f(0,2)=1$$

MIN. ASS. in $(1,2)$ e $(-1,2)$:

$$f(\pm 1,2)=-3$$

4)



$y=\sqrt{1-x^2}$ è la semicirconferenza superiore.

D è un quarto di cerchio.

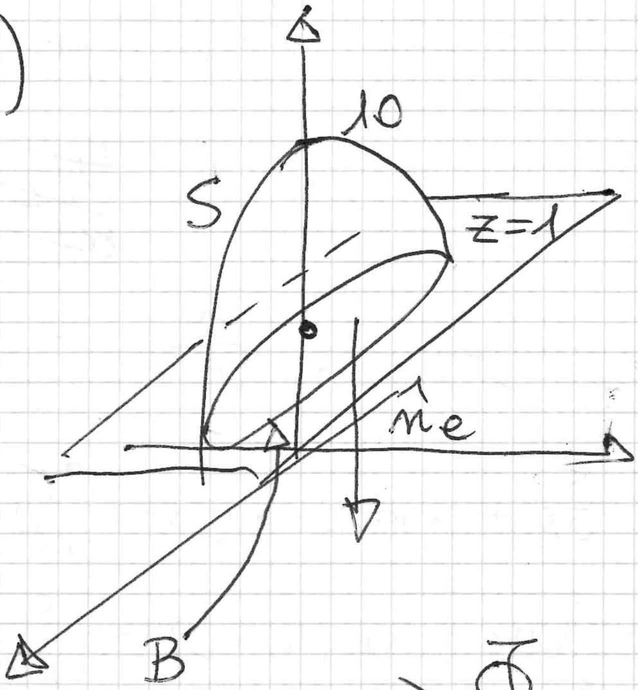
$$I=\iint_D x^2\,dx\,dy=\int_0^{\frac{\pi}{2}}d\vartheta\int_0^1\rho^3\cos^2\vartheta\,d\rho$$

$$=\int_0^{\frac{\pi}{2}}\cos^2\vartheta\,d\vartheta\int_0^1\rho^3\,d\rho$$

$$=\left[\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\vartheta}{2}\,d\vartheta\left[\frac{\rho^4}{4}\right]_0^1\right]$$

$$=\frac{1}{8}\left[\vartheta+\frac{1}{2}\sin 2\vartheta\right]_0^{\frac{\pi}{2}}=\frac{\pi}{16}.$$

5)



$$\operatorname{div}\vec{F} = 1-1 = 0$$

$$\Phi_S = \iiint_D \operatorname{div}\vec{F} - \Phi_B$$

dove

$$D = \left\{(x,y,z) : 1 \leq z \leq 10 - x^2 - 9y^2\right\}$$

$$\Rightarrow \Phi_S = -\Phi_B$$

$$\hat{n}_e\Big|_B = (0,0,-1) \qquad \vec{F}\cdot\hat{n}_e = -x\Big|_B$$

$$\text{Ma } B = \left\{(x,y,z) : z=1 \ ; \ x^2+9y^2 \leq 9\right\}$$

$$= \left\{(x,y,z) : z=1 ; \left(\frac{x}{3}\right)^2 + y^2 \leq 1\right\}$$

$$= \begin{cases} z = 1 \\ x = 3\rho\cos\vartheta & \rho\in[0,1] \\ y = \rho\sin\vartheta & \vartheta\in[0,2\pi] \end{cases}$$

$$\Rightarrow \Phi_S = -\Phi_B = -\iint_B \cancel{3\rho\cos\vartheta} -x \, dx\, dy$$

$$= \int_0^{2\pi} d\vartheta \int_0^1 \rho\cdot(3\rho)\cos\vartheta \, d\rho$$

$$= \underbrace{\int_0^{2\pi}\cos\vartheta\, d\vartheta}_{=0} \int_0^1 3\rho^2 d\rho = 0.$$

6) OMOGENEA ASSOCIATA:

$$\alpha^2-4=0 \quad \Rightarrow \quad \alpha_1=2\ ;\ \alpha_2=-2.$$

$$\Rightarrow y_0(x)=C_1e^{2x}+C_2e^{-2x}$$

Metodo di somiglianza: poichè $\alpha=2$ è radice del polinomio caratteristico, allora

$$y_p(x)=Axe^{2x}+B$$

$$y_p'=A(1+2x)e^{2x}$$

$$y_p''=A(4+4x)e^{2x}$$

$$\Rightarrow A(4+4x)e^{2x}-4Axe^{2x}-4B=e^{2x}-1$$

$$4Ae^{2x}(1+\cancel{x-x})-4B=e^{2x}-1$$

$$\begin{cases}4A=1\\4B=1\end{cases} \Rightarrow \begin{cases}A=\frac{1}{4}\\B=\frac{1}{4}\end{cases}$$

$$\Rightarrow y(x)=C_1e^{2x}+C_2e^{-2x}+\frac{1}{4}xe^{2x}+\frac{1}{4}.$$

$$y(-1)=C_1e^{-2}+C_2e^{2}-\frac{1}{4}e^{-2}+\frac{1}{4}=1$$

$$y'(x)=2C_1e^{2x}-2C_2e^{-2x}+\frac{1}{4}(1+2x)e^{2x}$$

$$y'(-1)=2C_1e^{-2}-2C_2e^{2}-\frac{1}{4}e^{-2}=0$$

$$\begin{cases} C_1 e^{-2} + C_2 e^{2} = \frac{1}{4} e^{-2} + \frac{3}{4} \\ C_1 e^{-2} - C_2 e^{2} = \frac{1}{8} e^{-2} \end{cases}$$

$$\begin{cases} 2C_1 e^{-2} = \frac{3}{8} e^{-2} + \frac{3}{4} \\ 2C_2 e^{2} = \frac{1}{8} e^{-2} + \frac{3}{4} \end{cases}$$

$$\begin{cases} C_1 = \frac{1}{2} e^{2} \left(\frac{3}{8} e^{-2} + \frac{3}{4} \right) = \frac{3}{16} + \frac{3}{8} e^{2} \\ C_2 = \frac{1}{2} e^{-2} \left(\frac{1}{8} e^{-2} + \frac{3}{4} \right) = \frac{1}{16} e^{-4} + \frac{3}{8} e^{-2} \end{cases}$$

$$\Rightarrow y(x) = \left(\frac{3}{16} + \frac{3}{8} e^{2} \right) e^{2x} + \left(\frac{1}{16} e^{-4} + \frac{3}{8} e^{-2} \right) e^{-2x} + \frac{1}{4} x e^{2x} + \frac{1}{4}.$$