SVOLGIMENTI PROVA SCRITTA di ANALISI 2 - 16/9/2026

1) Poiché $x \geq 0$, $\varphi_n(x)$ è sempre definito.
Quindi $D_c = \mathbb{R}_+ = [0, +\infty)$.

$x = 0$: $\varphi_n(0) = \dfrac{1}{n^2+1} \xrightarrow[n\to+\infty]{} 0$

$x > 0$: $\varphi_n(x) = \dfrac{nx+1}{n^2(x+1)+1} \sim \dfrac{nx}{n^2(x+1)} \xrightarrow[n\to+\infty]{} 0$
$\forall x > 0$.

$\Rightarrow \varphi_n(x) = 0 \quad \forall x \in [0, +\infty)$.

$0 < \varphi_n(x) < \dfrac{nx+n}{n^2(x+1)} = \dfrac{n(x+1)}{n^2(x+1)} = \dfrac{1}{n} \xrightarrow[n\to+\infty]{} 0$

$\Rightarrow D_{cu} = D_{cp} = [0, +\infty)$.

In alternativa:

$|\varphi_n(x) - \varphi(x)| = \varphi_n(x)$

$\Rightarrow |\varphi_n(x) - \varphi(x)|' = \varphi_n'(x) = \left[\dfrac{nx+1}{n^2(x+1)+1}\right]'$

$= \dfrac{n\left[n^2(x+1)+1\right] - (nx+1)n^2}{\left[n^2(x+1)+1\right]^2} = \dfrac{n^3x + n^3 + n - n^3x - n^2}{\left[\quad\right]^2}$

Poiché $x^3 - x^2 + x = x(x^2 - x + 1) \geq 0 \quad \forall x \geq 0$,

$\Rightarrow \varphi_n'(x) \geq 0 \quad \forall x \in D_{cp}$.

$$\Rightarrow \sup_{[0,+\infty)} |\varphi_n(x)-\varphi(x)| = \lim_{x\to+\infty} \varphi_n(x) = \lim_{x\to+\infty} \frac{nx}{n^2x} = \frac{1}{n}.$$

Quindi $\lim_{n\to+\infty} \sup_{[0,+\infty)} |\varphi_n(x)-\varphi(x)| = \lim_{n\to+\infty} \frac{1}{n} = 0.$

(2)

SERIE:

$$\forall x>0 \quad \varphi_n(x) \underset{n\to+\infty}{\sim} \frac{nx}{n^2(x+1)} = \frac{x}{(x+1)}\cdot\frac{1}{n}$$

e la serie ha lo stesso carattere, divergente, della serie armonica.

Per $x=0$ $\quad \sum \varphi_n(0) = \sum \frac{1}{n^2+1}$ convergente.

$$\Rightarrow \mathcal{D}_{cp} = \mathcal{D}_{CT} = \{0\}.$$

2) $$\left|\frac{e^{xy^2}-1}{2x^2+y^2}\right| \underset{(x,y)\to(0,0)}{\sim} \left|\frac{xy^2}{2x^2+y^2}\right| \le \left|\frac{xy^2}{y^2}\right| = |x| \to 0$$

$\Rightarrow f$ è continua in $(0,0)$.

$$f(x,0)=f(0,y)=0 \Rightarrow f_x(0,0)=f_y(0,0)=0.$$

$$\frac{df}{d\vec{v}}(0,0) = \lim_{t\to0} \frac{e^{t^3\alpha\beta^2}-1}{(2\alpha^2+\beta^2)t^3} = \lim_{t\to0} \frac{t^3\alpha\beta^2}{t^3(2\alpha^2+\beta^2)}$$

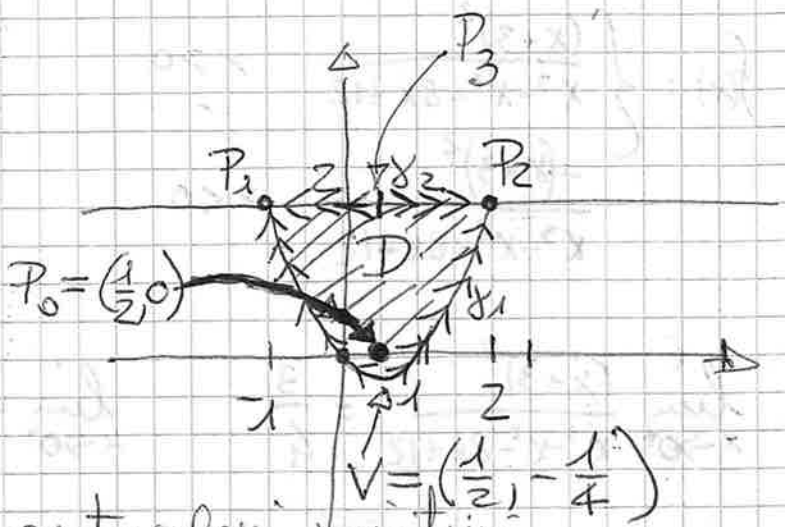
$$= \frac{\alpha\beta^2}{2\alpha^2+\beta^2} \qquad \forall \vec{v}$$

Poiché non vale la formula del gradiente, f NON è differenziabile in $(0,0)$.

3) In $\overset{\circ}{D}$:

$$\begin{cases} f_x = 2x-1 = 0 \iff x = \frac{1}{2} \\ f_y = \frac{4y}{1+y^2} \iff y = 0 \end{cases} ; \quad P_0 \equiv \left(\frac{1}{2}, 0\right) \in \overset{\circ}{D}$$

$$H_f(x,y) = \begin{pmatrix} 2 & 0 \\ 0 & \frac{4(1-y^2)}{(1+y^2)^2} \end{pmatrix}$$



$$\Rightarrow H_f\left(\frac{1}{2}, 0\right) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

autovalori positivi

$\Rightarrow P_0$ punto di MIN. REL.

$$f\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$$

Sia $\partial D = \gamma_1 \cup \gamma_2$

oppure $\begin{cases} x^2 - x = y \\ y \in \left[-\frac{1}{4}, 2\right] \end{cases}$

con γ_1: $\begin{cases} y = x^2 - x \\ x \in [-1, 2] \end{cases}$; γ_2: $\begin{cases} y = 2 \\ x \in [-1, 2] \end{cases}$

Ponendo su γ_1 $x^2 - x = y$, abbiamo

$$f(x^2 - x = y) = y + 2\ln(1+y^2)$$

$$f\Big|'_{\gamma_1} = 1 + \frac{4y}{1+y^2} = \frac{y^2+4y+1}{1+y^2}$$

Si osserva che $y^2+4y+1 > 0$ se $y_1 < -2-\sqrt{3}$ $\vee$ $y_2 > -2+\sqrt{3} \cong -0{,}27$

mentre $y_V = -\frac{1}{4} = -0{,}25 > y_2$,

pertanto $f'|_{\gamma_1} > 0 \quad \forall y \in \left[-\frac{1}{4}, 2\right]$ (4)

Si osservi che, al variare di y, la parabola va percorsa lungo due rami differenti.

$$f|_{\gamma_2} = x^2 - x + 2\ln 5$$

$$\left(f|_{\gamma_2}\right)' = 2x - 1 > 0 \iff x > \frac{1}{2}$$

I punti da considerare, vedendo il grafico, sono

$P_1 = (-1, 2)$ di MAX. REL. VINCOLATO

$P_2 = (1, 2)$ di MAX. REL. "

$P_3 = \left(\frac{1}{2}, 2\right)$ di MIN. REL. "

$V = \left(\frac{1}{2}, -\frac{1}{4}\right)$ di MIN. REL. "

$P_0 = \left(\frac{1}{2}, 0\right)$ di MIN. REL.

$f(P_1) = 2 + 2\ln 5$

$f(P_2) = 2 + 2\ln 5$ } MAX. ASS.

$f(P_3) = -\frac{1}{4} + 2\ln 5$

$f(V) = -\frac{1}{4} + 2\ln\left(\frac{17}{16}\right)$

$f(P_0) = -\frac{1}{4}$ MIN. ASS.

4) $\vec{F}$ è definito su tutto $\mathbb{R}^2$.

$\vec{F}$ è irrotazionale:

$$X_y = \cancel{xy} \cdot -\frac{2x}{1+x^2} + z = Y_x$$

$$X_z = \frac{2z}{1+z^2} + y = Z_x$$

$$Y_z = x = Z_y$$

$\Rightarrow \vec{F}$ è conservativo.

a) Primo metodo: determiniamo un potenziale

$$U(x,y,z) = \int Y(x,y,z)\,dy =$$

$$xyz - y\ln(1+x^2) + \varphi(x,z)$$

$$\frac{\partial U}{\partial x} = yz - \frac{2xy}{1+x^2} + \frac{\partial \varphi}{\partial x} = X = \ln(1+z^2) - \frac{2xy}{1+x^2} + yz$$

$$\Rightarrow \frac{\partial \varphi}{\partial x} = \ln(1+z^2) \Rightarrow \varphi(x,z) = x\ln(1+z^2) + \psi(z)$$

$$\Rightarrow U(x,y,z) = xyz - y\ln(1+x^2) + x\ln(1+z^2) + \psi(z)$$

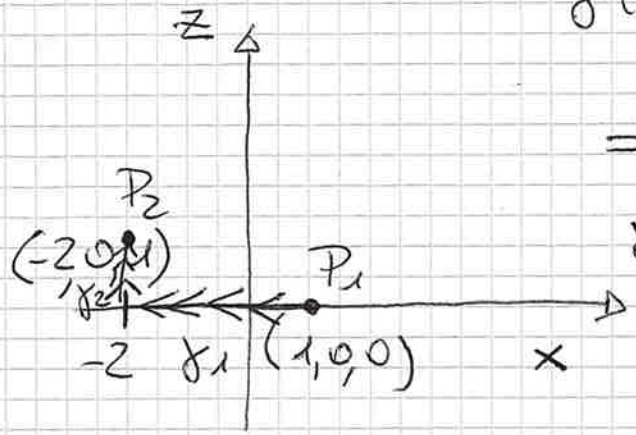
$$\frac{\partial U}{\partial z} = xy + \underbrace{\cancel{\ln(1+z^2)} + \frac{2xz}{1+z^2}} + \psi'(z) = Z = xy + \frac{2xz}{1+z^2}$$

$$\Rightarrow \psi'(z) = 0$$

$\Rightarrow U(x,y,z) = xyz - y\ln(1+x^2) + x\ln(1+z^2) + C$

La curva è tale che $\gamma(0) = (+1,0,0) = P_1$

$\gamma(1) = (-2,0,1) = P_2$



$\Rightarrow \int_{\gamma} \vec{F}\cdot d\vec{s}$

$= U(-2,0,1) - U(1,0,0) = -2\ln 2$.

b) Secondo metodo: poichè $\vec{F}$ è conservativo, andiamo da P_1 a P_2 lungo $\gamma_1 \cup \gamma_2$, dove

$\gamma_1^- : \begin{cases} y=0 \\ z=0 \\ x\in[-2;1] \end{cases}$ ($dy=0$, $dz=0$) ; $\gamma_2^+ = \begin{cases} x=-2 \\ y=0 \\ z\in[0,1] \end{cases}$ ($dx=0$, $dy=0$)

$$\Rightarrow \int_{\gamma} \vec{F}\cdot d\vec{s} = -\int_{\gamma_1} \vec{F}\cdot d\vec{s} + \int_{\gamma_2} \vec{F}\cdot d\vec{s}$$

$$= -\int_{-2}^{1} 0\,dx + \int_{0}^{1} \frac{-4z}{1+z^2}\,dz = -2\ln(1+z^2)\Big|_0^1 = -2\ln 2.$$

5) La superficie ha equazione

$$\Sigma: \begin{cases} x = t\cos\vartheta \\ y = t\sin\vartheta \\ z = \cosh t \end{cases} \qquad t\in[0,1], \quad \vartheta\in[0,\pi].$$

La jacobiana è

$$\begin{pmatrix} \cos\vartheta & \sin\vartheta & \sinh t \\ -t\sin\vartheta & t\cos\vartheta & 0 \end{pmatrix}$$

$$\Rightarrow L = -t\cos\vartheta \sinh t$$

$$M = -t\sin\vartheta \sinh t$$

$$N = t$$

$$\Rightarrow W(t,\vartheta) = \sqrt{t^2\sinh^2 t + t^2} = |t|\sqrt{\sinh^2 t + 1}$$

$$= t|\cosh t| = t\cosh t$$

$$\Rightarrow \text{Area}\,\Sigma = \int_0^{\pi} d\vartheta \int_0^1 t\cosh t\, dt =$$

$$= \pi\left[t\sinh t\Big|_0^1 - \int_0^1 \sinh t\, dt\right] =$$

$$= \pi\left[\sinh 1 - \cosh t\Big|_0^1\right] = \pi\left[\sinh 1 - \cosh 1 + 1\right]$$

$$= \pi\left[\frac{e^1 - e^{-1}}{2} - \frac{e^1 - e^{-1}}{2} + 1\right] = \pi\left(1 - \frac{1}{e}\right)$$

6) OMOGENEA:

$\alpha^2 + 1 = 0 \Rightarrow y_0(x) = C_1 \cos x + C_2 \sin x$

$\alpha = \pm i$

NON OMOGENEA: $y_p(x) = Ae^{-x}$

$y_p' = -Ae^{-x}$; $y_p'' = Ae^{-x}$

$\Rightarrow 2Ae^{-x} = e^{-x} \Rightarrow A = \frac{1}{2}$

$\Rightarrow y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2} e^{-x}$

$y(0) = C_1 + \frac{1}{2} = 1 \Rightarrow C_1 = \frac{1}{2}$

$y'(x) = -\frac{1}{2} \sin x + C_2 \cos x - \frac{1}{2} e^{-x}$

$y'(0) = C_2 - \frac{1}{2} = 0 \Rightarrow C_1 = C_2 = \frac{1}{2}$

$\Rightarrow y(x) = \frac{1}{2} \left[\cos x + \sin x + e^{-x} \right]$.