JOURNAL OF THE COURSE INTRODUCTION TO SOBOLEV SPACES AND DIFFERENTIAL EQUATIONS A.A. 2015/2016 CORSO DI DOTTORATO IN MODELLI MATEMATICI PER L'INGEGNERIA, ELETTROMAGNETISMO E NANOSCIENZE

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Results listed in *this font* have been proved in detail.
References:

- *DiBenedetto*: E.DiBenedetto, Real Analysis, Birkhauser.
- *Brezis*: H. Brézis, Analisi Funzionale, Liguori.

1. Monday 2016-03-07

Introduction to the course.
Lebesgue outer measure and measure in $\mathbb{R}^N$.
Borel σ-algebra; Cantor's ternary set.
Measurable functions. *Approximation by simple functions.* Lebesgue integral.
Convergence theorems (Beppo Levi, Fatou, Lebesgue).
The theorems of Fubini and Tonelli. A counterexample.

Reference sections in textbooks: DiBenedetto 9.7, Brezis Appendix 1-6.

2. Wednesday 2016-03-09

Definition of $L^p(\Omega)$, $1 \le p \le \infty$.
The spaces $L^p(\Omega)$ are normed vector spaces. Hölder and Minkowski inequalities.
$L^p(\Omega)$ is complete for all $1 \le p \le \infty$. From any converging sequence in $L^p(\Omega)$ one may extract a subsequence converging a.e..
The smooth functions with compact support are dense in $L^p(\Omega)$ of Ω is open and $p < \infty$.
Compactness criterion in $L^p(\Omega)$, $p < \infty$ and $|\Omega| < \infty$.
Linear bounded functionals. All bounded linear functionals in $L^p(\Omega)$, $p < \infty$, are represented by functions in $L^{p'}(\Omega)$. Separability of $L^p(\Omega)$, $p < \infty$.
Weak convergence. Bound for the norm of the weak limit. Convergence follows from weak convergence and convergence of the norms, if $1 < p < \infty$. Products of weakly converging and strongly converging sequences. Bounded sets in $L^p(\Omega)$ are weakly pre-compact if $1 < p < \infty$.
Examples: *1) a sequence bounded in $L^1(\mathbb{R}^N)$ but without weakly converging subsequences. 2) An example where weak convergence can not be taken inside a nonlinear function. 3) An example of $g \in L^p(\Omega)$, $p < \infty$, which is unbounded in any open set.*

Reference sections in textbooks: Brezis, chapter 4.

3. Monday 2016-03-14

Definition of Sobolev space $W_1^p(\Omega)$. Completeness, reflexivity and separability of Sobolev spaces.
Examples of Sobolev functions. *The case of $N = 1$. The case of $u(x) = |x|^{-\alpha}$.*
Approximation by smooth functions; general Ω and smooth Ω.
Three equivalent formulations for $u \in W_1^p(\Omega)$: definition, via Riesz representation theorem, via Lipschitz continuity in the L^p norm of the translation operator.
The extension operator from $W_1^p(\Omega)$ to $W_1^p(\mathbb{R}^N)$.
Some properties of Sobolev functions: functions with zero gradient, Leibniz rule, chain rule, integration along (almost every) straight line parallel to an axis.
Sobolev embedding theorem ($p < N$).
Rellich-Kondrachov compactness theorem.

Reference sections in textbooks: Brezis, chapter 9.

4. Tuesday 2016-03-22

Definition of the space $\overset{\circ}{W}{}^{p}_{1}(\Omega)$, $1 \leq p < +\infty$.
Characterization of $C(\overline{\Omega}) \cap \overset{\circ}{W}{}^{p}_{1}(\Omega)$.
Characterization of $\overset{\circ}{W}{}^{p}_{1}(\Omega)$ by means of test functions and of trivial extension to $\mathbb{R}^N$.
Poincaré inequality. Equivalence of the gradient norm in $\overset{\circ}{W}{}^{p}_{1}(\Omega)$.
Representation of functionals in the dual space of $\overset{\circ}{W}{}^{p}_{1}(\Omega)$.
The elliptic Dirichlet boundary value problem for $f \in L^2(\Omega)$

$$-\Delta u + u = f\,, \quad \text{in } \Omega; \qquad u = 0\,, \quad \text{on } \partial\Omega.$$

Classical solutions and weak solutions. *A classical solution is also a weak solution. A weak solution, if it exists, is unique.*
Hilbert spaces. Cauchy-Schwarz inequality and parallelogram law.
Let K be a closed non-empty convex set of the Hilbert space H. *Existence of the minimum on K of the distance from a given $f \in H$. Characterization of the point of minimum in terms of the scalar product. Uniqueness of the point of minimum.* Lipschitz continuity of the projection. Characterization of the minimum when K is a subspace.
Theorem of representation of linear bounded functionals on a Hilbert space (Riesz-Frechet).
Examples of Hilbert spaces: $L^2(\Omega)$, $W^2_1(\Omega)$. *Application to the boundary value problem.*

Reference sections in textbooks: Brezis, sections 5.1–5.2, 9.4–9.5.

5. Thursday 2016-03-31

Traces of functions in $W_1^p(\Omega)$, $1 \le p < +\infty$. *Trace inequality: there exists a constant $C = C(\Omega, \varepsilon, p)$ such that for all $\varepsilon > 0$ and $u \in C^1(\overline{\Omega})$*

$$\|u\|_{p,\Omega} \le \varepsilon^{p-1} \|\nabla u\|_{p,\Omega} + C(\Omega, \varepsilon, p) \|u\|_{p,\Omega} \,.$$

Definition of the trace operator; its continuity and compactness if $p > 1$.

The space of functions with zero trace is $W_{o1}^p(\Omega)$. Green's identity.

Assume that u_n converges weakly to u in $W_1^p(\Omega)$, $p > 1$. Then the trace of u_n converges weakly to the trace of u in $L^p(\partial\Omega)$.

Definition of weak solutions of the problem

$$-(a_{ij}(x) u_{x_i})_{x_j} = 0 \,, \quad \text{in } \Omega; \qquad u = g \,, \quad \text{on } \partial\Omega,$$

for $g \in W_1^p(\Omega)$. *Existence and uniqueness of weak solutions via reduction to the problem for $v = u - g$*

$$-(a_{ij}(x) v_{x_i})_{x_j} = (a_{ij}(x) g_{x_i})_{x_j} \,, \quad \text{in } \Omega; \qquad v = 0 \,, \quad \text{on } \partial\Omega.$$

Use of the scalar product

$$\int_\Omega a_{ij}(x) \varphi_{x_i} v_{x_j} \,\mathrm{d}x \,.$$

If $g_1 = g_2$ on $\partial\Omega$ then $u_1 = u_2$.

The Neumann problem

$$-(a_{ij}(x) u_{x_i})_{x_j} = f \,, \quad \text{in } \Omega; \qquad a_{ij} u_{x_i} \nu_j = \psi \,, \quad \text{on } \partial\Omega.$$

Definition of weak solutions. The necessary condition for the existence of solutions. *Existence of solutions via Riesz-Frechet theorem applied in*

$$\hat{H} = \left\{ u \in W_1^2(\Omega) \mid \int_\Omega u \,\mathrm{d}x = 0 \right\}.$$

Poincaré inequality: there exists a $C = C(\Omega) > 0$ such that

$$\|u\|_{2,\Omega} \le C \|\nabla u\|_{2,\Omega} \,, \qquad u \in \hat{H} \,.$$

Use of the necessary condition to recover a solution in the full sense.

Reference sections in textbooks: Brezis, chapter 9.

6. Monday 2016-04-04

Continuous and coercive bilinear form on a Hilbert space H.
Stampacchia theorem. Non-symmetric and symmetric forms. Lax-Milgram theorem.
Example of projection from $L^2(\Omega)$ to $K = \{u \geq f\} \cap L^2(\Omega)$.
The obstacle problem; *existence and uniqueness of a solution via Stampacchia theorem.* Interpretation of the obstacle problem as a projection problem.

Reference sections in textbooks: Brezis chapter 5.3.