JOURNAL OF THE COURSE PARTIAL DIFFERENTIAL EQUATIONS OF PARABOLIC TYPE A.A. 2020–2021 CORSO DI DOTTORATO IN MODELLI MATEMATICI PER L'INGEGNERIA, ELETTROMAGNETISMO E NANOSCIENZE

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DIP. SCIENZE DI BASE E APPLICATE PER L'INGEGNERIA
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Results listed *in this font* have been proved in detail. References:

- DB: E.DiBenedetto, Partial Differential Equations, Birkhäuser.
- LE: L.Evans, Partial Differential Equations, AMS.
- SP: D.Andreucci, Lecture Notes on Free Boundary Problems for Parabolic Equations
- FM: D.Andreucci, Appunti per il corso di Fisica Matematica

1. Friday 19/03/2021
(10-12 Online)

Presentation of the course.
Derivation of the diffusion (or heat) equation.
Definitions of parabolic interior, parabolic boundary, classical solution to a linear parabolic equation.
Maximum principle for classical solutions.
Remarks on: equations with term cu; various boundary conditions (Dirichlet, Neumann, Robin).

Homework 1.1. 1) Show how to use the maximum principle to obtain bounds for solutions to problems with Neumann or Robin boundary conditions.
2) Given that the fundamental solution Γ of the heat equation is written as

$$\Gamma(x,t) = \frac{1}{R(t)^N} \varphi\Big(\frac{|x|}{R(t)}\Big) ,$$

find R and φ. □

Reference sections in textbooks: SP: A.1; FM: 11.4.2.

2. Tuesday 23/03/2021
(10:30-12:30 Online)

Strong maximum principle. Parabolic Hopf Lemma. Application to uniqueness theorems for solutions to initial value boundary problems. The Cauchy problem for the heat equation. The fundamental solution and the representation formula for other solutions (in the admissible growth class). Sketch of proof of differentiability under the integral sign.
A counterexample by Tychonov. Properties of non-negative solutions.

Homework 2.1. 1) Find $c \in \boldsymbol{R}$ such that $u(x,t) \to c$, $t \to +\infty$ for every fixed $x \in \boldsymbol{R}$, where u solves

$$\begin{aligned} u_t - u_{xx} &= 0\,, & x \in \boldsymbol{R}\,, t > 0\,, \\ u(x,0) &= \chi_I(x)\,, & x \in \boldsymbol{R}\,, \end{aligned}$$

and $I = \cup_{n \in \boldsymbol{Z}} (2n, 2n+1)$.
2) Find the optimal $f(U,t)$ such that $|u(x,t)| \le f(U,t)$ for $t > 0$, if u is the solution of the Cauchy problem in $\boldsymbol{R}^N$ with $u_0 \in L^1(\boldsymbol{R}^N)$; here $U = \|u_0\|_{L^1(\boldsymbol{R}^N)}$. □

Reference sections in textbooks: SP: A.2, A.3; DB: 5.2, 5.5; LE: 2.3.1; FM: 11.4, 11.5.

3. Friday 26/03/2021
(09-11 Online)

Non-negative solutions to the heat equation in the strip $S_T = \boldsymbol{R}^N \times (0,T)$.

Maximum principle in S_T under the assumption $u(x,t) \le Ce^{K|x|^2}$.

Theorem 3.1. *Il $u \ge 0$ solves $u_t - \Delta u = 0$ in S_T, then*

$$u(x,t) \ge \int_{\boldsymbol{R}^N} \Gamma(x-y,t-\tau)u(y,\tau)\,\mathrm{d}y\,,$$

for $x \in \boldsymbol{R}^N$, $T > t > \tau > 0$.

Lemma: minimum principle for solutions to the heat equations not continuous at $t=0$, but bounded.

Theorem 3.2. *If $u \ge 0$ solves $u_t - \Delta u = 0$ in S_T, then*

$$u(x,t) = \int_{\boldsymbol{R}^N} \Gamma(x-y,t-\tau)u(y,\tau)\,\mathrm{d}y\,,$$

for $x \in \boldsymbol{R}^N$, $T > t > \tau > 0$.

Theorem 3.3. *If $u \ge 0$ solves $u_t - \Delta u = 0$ in S_T, then there exists a unique Radon measure μ in $\boldsymbol{R}^N$ such that*

$$u(x,t) = \int_{\boldsymbol{R}^N} \Gamma(x-y,t)\,\mathrm{d}\mu\,,$$

for $x \in \boldsymbol{R}^N$, $T > t > 0$.

Property of infinite speed of propagation for solutions to the heat equation (but not to degenerate parabolic equations).

Homework 3.4. Spread of mass: prove that if $u \ge 0$ and $\operatorname{supp} u(x,0) \subset B_R$ then for every $\varepsilon \in (0,1)$ there exists C_ε such that

$$\int_{|x| \le R + C_\varepsilon \sqrt{t}} u(x,t)\,\mathrm{d}x \ge (1-\varepsilon) \int_{B_R} u(x,0)\,\mathrm{d}x\,, \qquad t > 0\,.$$

□

Reference sections in textbooks: DB: 5.4, 5.14; FM: 11.5.

4. Tuesday 30/03/2021
(10:30-12:30 Online)

Theorem of uniqueness of solutions to the Cauchy problem for the heat equation, taking data in the sense of $L^1_{\mathrm{loc}}(\boldsymbol{R}^N)$ and such that for all (x,t) and $\varepsilon > 0$

$$\sup_{0<\tau<t-\varepsilon} \int_{\boldsymbol{R}^N} |u(y,\tau)| \Gamma(x-y,t-\tau)\,\mathrm{d}y < +\infty\,.$$

Problems of divergence type with non-smooth coefficients and/or data. The basic energy estimates, and existence of weak solutions in a suitable sense by approximation (assuming existence of approximating smooth solutions). The problem of uniqueness.

Homework 4.1. Prove that if u, ∇u, $u_t \in L^2(\Omega_T)$, and for $u_0 \in L^2(\Omega)$ we have

$$\iint_{\Omega_T} \{-u\varphi_t + \nabla u \nabla \varphi\}\,\mathrm{d}x\,\mathrm{d}t = \int_{\Omega} u_0(x)\varphi(x,0)\,\mathrm{d}x\,,$$

for all φ regular as u, with $\varphi = 0$ on $t = T$ and $\partial\Omega$, then $u(x,0) = u_0(x)$ in the sense of traces. □

Reference sections in textbooks: DB: 5.14; LE: 7.1; FM: 19.3.

5. Friday 09/04/2021
(10:30-12:30 Online)

Existence theorem for the initial value boundary problem for a nonlinear uniformly parabolic equation (with zero Dirichlet data): the Galerkin method. Approximation, compactness, limit (identification of the diffusion term postponed).

Barenblatt-Pattle solutions for the porous media and p-laplacian equations (with compact support and with critical growth as $|x| \to +\infty$).

Homework 5.1. 1) Lemmas used in the proof of Galerkin method: uniform approximation in a Sobolev space by means an orthonormal complete system in L^2; existence of smooth orthonormal systems in L^2, dense in a Sobolev space.
2) Investigation of the properties of Barenblatt-Pattle solutions. □

Reference sections in textbooks: LE: 7.1.1, 7.1.2; notes on Classroom.

6. Tuesday 13/04/2021
(10:30-12:30 Online)

Identification of the nonlinear diffusion term in the limit of Galerkin's approximation procedure: Minty's trick. Steklov averages.
Weak maximum principle for weak solutions.
Comments on the structure of Barenblatt-Pattle solutions and connections with the nonlinearity in the equation.

Reference sections in textbooks: LE: 7.1.1, 7.1.2; notes on Classroom.

7. Friday 16/04/2021
(10:30-12:30 Online)

Caccioppoli inequality for cuts of local solutions.
Local estimates for nonnegative solutions of the porous media equation: the $L^1 - L^2$ estimate.
Asymptotic decay for large times of solutions to Dirichlet problems in bounded domains with zero boundary data, for the heat equation and for the porous media equation.

Homework 7.1. Discuss the blow up of solutions to

$$\begin{aligned} u_t - \Delta u &= u^p\,, && \text{in } \Omega \times (0,T);\\ u(x,t) &= 0\,, && \text{on } \partial\Omega \times (0,T);\\ u(x,0) &= u_0(x) \geq 0\,, && x \in \Omega\,, \end{aligned}$$

where $p > 1$, using the first eigenfunction $\Delta w_1 = -\lambda_1 w_1$ in Ω, $w_1 = 0$ on $\partial\Omega$. □

Reference sections in textbooks: DB: 10.1, 10.2 (somehow related); Notes on Classroom.

8. Tuesday 20/04/2021
(11-13 Online)

Local estimates for nonnegative solutions of the porous media equation: the $L^1 - L^\infty$ estimate.
Theorem of existence of the Cauchy problem for the porous media equation, under the assumption

$$\sup_{\rho > r} \rho^{-N - \frac{2}{m-1}} \int\limits_{B_\rho(0)} |u_0(x)| \,\mathrm{d}x < +\infty\,.$$

Qualitative estimates.

Reference sections in textbooks: Notes on Classroom.

9. Friday 23/04/2021
(10:30-12:30 Online)

The Cauchy problem for the porous media equation for initial data with finite mass. Global existence.
The property of finite speed of propagation.
Problems for $u_t - \Delta u^m = u^p$, $p > 1$*: non-existence of global solutions if* $p < m + 2/N$.

Reference sections in textbooks: Notes on Classroom.

10. Tuesday 27/04/2021
(10:30-12:30 Online)

Problems for $u_t - \Delta u^m = u^p$, $p > 1$*: necessity of conditions on the initial data. Existence of global solutions in the case* $p > m + 2/N$.
Log-convexity of $\|u(t)\|^2_{L^2(\Omega)}$ *if* u *solves a Dirichlet problem for the heat equation with zero boundary data.*
The Stefan problem and condition.

Reference sections in textbooks: DB: 5.11; SP: 1.1, 1.2.

11. Friday 30/04/2021
(10:30–12:30 Online)

Weak formulation(s) of the Stefan problem; continuous dependence on the initial data; existence in the case of source depending on temperature.

Homework 11.1. Prove that a weak solution of the Stefan problem, such that the region of critical temperature is a smooth surface S, and the temperature is smooth on either side of S (up to S) satisfies the Stefan condition. □

Reference sections in textbooks: SP: 2.1-2.5.

12. Tuesday 04/05/2021
(10:30–12:30 Online)

Final seminars:

(1) Asymptotic behavior in bounded domains, linear problems (notes of Ravello 2003, Sections 1–2).
(2) Asymptotic behavior in bounded domains, nonlinear problems (notes of Ravello 2003, Sections 3–5).
(3) Nonhomogeneous problems and mean value formula for the heat equation (LE, Sections 2.3.1.c–2.3.3.a).
(4) Harnack inequality and the strong maximum principle (LE, Sections 7.1.4.b–7.1.4.c).
(5) Semigroup theory (LE, Section 7.4).
(6) Classical formulation of the Stefan problem (SP, Chapter 1).
(7) Relaxed formulation of the Stefan problem (SP, Chapter 3 and Appendix D).

Remarks on the seminars.

Homework 12.1. Discuss the problem of the critical length for $u_t - u_{xx} = cu$, $c > 0$. $\square$

End of the course