Es: Sia $t \in \mathbb{R}$.

$$U_t := \left\langle \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix}, \begin{pmatrix} t \\ t \\ t \\ t \end{pmatrix} \right\rangle ; \quad W_t := \left\langle \begin{pmatrix} t-2 \\ -t \\ -3t \\ t \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle \subset \mathbb{R}^4$$

1) Esiste $t \in \mathbb{R}$ tale che $U_t + W_t = \mathbb{R}^4$?

2) Per quali $t \in \mathbb{R}$ si ha che $\dim(U_t \cap W_t) = 1$?

3) Determinare una base di $U_1 \cap W_1$ ed estenderla ad una base di $\mathbb{R}^4$.

Sol.: Se $t = 0$: $\quad U_0 = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \quad W_0 = \left\langle \begin{pmatrix} -2 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle.$

Quindi $U_0 = W_0 = \langle e_1 \rangle$. $\quad \dim U_0 \cap W_0 = 1$

Se $t \neq 0$

$$U_t = \left\langle \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad W_t = \left\langle \begin{pmatrix} t-2 \\ -t \\ -3t \\ t \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} t \\ 0 \\ -t \\ \cancel{-}t \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle$$

$$\overset{t \neq 0}{=} \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle$$

$t \neq 0$

$$U_t = \left\langle \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad W_t = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle$$

$$\Rightarrow \dim U_t = 2, \quad \dim W_t = 2$$

Formula di Grassmann

$$2 \leq \dim(U_t + W_t) \leq 4, \qquad \dim(U_t + W_t) + \dim(U_t \cap W_t) = 4$$

$\dim U_t + W_t$	2	3	4
$\dim U_t \cap W_t$	2	1	0
		?	?

Formula di Grassmann:

$$\dim(U+W) + \dim(U \cap W) = \dim U + \dim W.$$

$t \neq 0$

$$U_t = \left\langle \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle, \quad W_t = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle$$

$$U_t + W_t = \mathbb{R}^4 . \ni$$

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Longleftrightarrow \left.\begin{array}{l} \cancel{1} = t_1 + \cancel{1} \\ 0 = t t_1 + 1 \\ -1 = 2t t_1 + 1 \\ t_2 = 1 \end{array}\right\} \quad \underline{NO}.$$

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = e_4 = t_1 \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + t_3 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + t_4 \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix}$$

$$\begin{cases} t_2 + t_3 = 1 \\ t_1 + 1 + 2t_4 = 0 \\ t t_1 + t_2 + t t_4 = 0 \\ 2t t_1 + t_2 - t_3 + 2t t_4 = 0 \end{cases}$$

$t \neq 0$

$$U_t = \left\langle \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle , \quad W_t = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle$$

$v \in U_t \cap W_t$

$$v = t_1 \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + s_2 \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix}$$

$$t_1 \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - s_1 \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} - s_2 \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} t_1 + \cancel{(t_2 - s_1)} - 2s_2 = 0 \\ t t_1 + t_2 - t s_2 = 0 \\ 2t t_1 + t_2 + s_1 - 2t s_2 = 0 \\ t_2 - s_1 = 0 \end{cases} \qquad \begin{cases} t_2 = s_1 \\ t_1 = 2s_2 \\ 2t s_2 + s_1 - t s_2 = 0 \\ 4t s_2 + s_1 + s_1 - 2t s_2 = 0 \end{cases}$$

$$\begin{cases} t_2 = s_1 \\ t_1 = 2s_2 \\ 2ts_2 + s_1 - ts_2 = 0 \\ 4ts_2 + s_1 + s_1 - 2ts_2 = 0 \end{cases} \qquad \begin{cases} t_2 = s_1 \\ t_1 = 2s_2 \\ ts_2 + s_1 = 0 \\ 2ts_2 + 2s_1 = 0 \end{cases} \qquad \boxed{\begin{matrix} t_2 = -ts_2 \\ t_1 = 2s_2 \\ s_1 = -ts_2 \end{matrix}}$$

Quindi possiamo trovare la soluzione per $s_2 = 1$

$$t_2 = -t$$

$$t_1 = 2$$

$$s_1 = -t$$

$$s_2 = 1$$

Verifichiamo

$$v = 2\begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix} - t\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \overset{?}{=} -t\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \quad \checkmark$$

$\Rightarrow \dim U_t \cap W_t \geq 1 \quad \forall t \Rightarrow$

Non ci sono $t \in \mathbb{R}$ t.c. $U_t + W_t = \mathbb{R}^4$.

$$\begin{tikzcd}[column sep=small]
 & U_t + W_t & \\
U_t \arrow[ur, phantom, sloped, "\subseteq"] & & W_t \arrow[ul, phantom, sloped, "\supseteq"] \\
 & U_t \cap W_t \arrow[ul, phantom, sloped, "\subseteq"] \arrow[ur, phantom, sloped, "\subseteq"] &
\end{tikzcd}$$

$$1 \leq \dim\left(U_t \cap W_t\right) \leq \min\left(\dim U_t, \dim W_t\right) = 2$$

$$\dim U_t \cap W_t = 2 \iff U_t = W_t.$$

Per quali t $\quad U_t = W_t$?

$t \neq 0$

$$U_t = \left\langle \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle , \quad W_t = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ t \\ 2t \\ 0 \end{pmatrix} \right\rangle$$

$U_t = W_t$?

$$\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = t_1 \begin{pmatrix} 1 \\ t \\ 2t \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \iff \begin{cases} 1 = t_1 + t_2 \\ 0 = t t_1 + t_2 \\ -1 = 2t t_1 + t_2 \\ \cancel{t} = t_2 \end{cases}$$

$$\begin{cases} 1 = t_1 + 1 & \to \quad t_1 = 0 \\ 0 = t t_1 + 1 & \to \quad t0 + 1 = 0 \quad \text{assurdo.} \\ -1 = 2t t_1 + 1 \\ 1 = t_2 \end{cases}$$

$$\Rightarrow \quad \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} \notin U_t \quad \forall t \neq 0 \quad \Rightarrow \quad U_t \neq W_t \quad \forall t \neq 0$$

$$\Rightarrow \quad \dim (U_t \cap W_t) = 1 \quad \forall t \neq 0$$

1) Mai

2) $\dim U_t \cap W_t = 1 \quad \forall t \in \mathbb{R}.$

3) $U_1 = \left\langle \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\rangle \qquad W_1 = \left\langle \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix} \right\rangle$

Base per $U_1 \cap W_1$.

$$v = 2 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix} = - \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 2 \\ 0 \end{pmatrix}$$

Dato che $v \neq 0_{\mathbb{R}^4}$, $\{v\}$ è una base di $U_1 \cap W_1$.

Usiamo l'algoritmo di generazione di basi:

$$\left\langle v = \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix} \right\rangle \not\ni \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = e_1$$

$\Rightarrow \quad \{ v, e_1 \}$ è lin. ind.

$$\langle v, e_1 \rangle \neq \mathbb{R}^4 , \quad \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\rangle \not\ni \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = e_2$$

$\Rightarrow \quad \{ v, e_1, e_2 \}$ è lin. Ind.

$$\langle v, e_1, e_2 \rangle \neq \mathbb{R}^4 , \quad \left\langle \begin{pmatrix} 1 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle \not\ni \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = e_3$$

$\Rightarrow \quad \{ v, e_1, e_2, e_3 \}$ è lin. Ind.

$\langle v, e_1, e_2, e_3 \rangle = \mathbb{R}^4$. Quindi $\{ v, e_1, e_2, e_3 \}$ è una base di $\mathbb{R}^4$ che estende una base di $U_1 \cap W_1$.

Es : Siano $\tau = \{ p_0, p_1, \dots, p_k \}$ polinomi tali che

$$gr(p_i) = i \qquad \forall i = 0, \dots, k.$$

Dimostrare che τ è lin. Ind.

$\left(\text{Es} \quad \tau = \{ 2, 3+2x, x+x^2, \dots \} \right)$.

Sol. :

$$t_0 p_0 + t_1 p_1 + \dots + t_k p_k = 0 + 0x + 0x^2 + \dots + 0x^k$$

$$t_k = 0$$

$$t_{k-1} + * t_k = 0$$

$$t_{k-2} + * t_{k-1} + * t_k = 0 \qquad \Rightarrow t_0 = \dots = t_k = 0.$$

$$\vdots$$

$$t_0 + * t_1 \dots + * t_k = 0$$

Es: $\tau = \{ \overset{p_0}{\overset{\parallel}{2}}, \overset{p_1}{\overset{\parallel}{2+2x}}, \overset{p_2}{\overset{\parallel}{x+3x^2}} \} \subset \mathbb{R}[x]$

$$t_0(2) + t_1(2+2x) + t_2(x+3x^2) = 0 + 0x + 0x^2$$

$$\begin{cases} 2t_0 + 2t_1 = 0 & t_0 = 0 \\ 2t_1 + t_2 = 0 & t_1 = 0 \\ 3t_2 = 0 \rightsquigarrow & t_2 = 0 \end{cases}$$

<u>OSS</u>: Questo vale se $p_0, \dots, p_k$ hanno gradi distinti.

Es: Sia $a \in \mathbb{R}$.

$$\mathcal{T} = \{1, x-a, (x-a)^2, (x-a)^3, \dots, (x-a)^n\}$$

è una base di $\mathbb{R}[x]_{\leq n}$. Quindi $\forall p \in \mathbb{R}[x]_{\leq n}$ $\exists!\ b_0, \dots, b_n \in \mathbb{R}$ t.c.

$$p(x) = b_0 + b_1 (x-a) + b_2 (x-a)^2 + \dots + b_n (x-a)^n \qquad (*)$$

$$b_0 = p(a) \qquad b_k = \frac{\left(\frac{d}{dx}\right)^k p(x)\big|_{x=a}}{k!} \qquad \frac{d}{dx} x^n = n x^{n-1}$$

$\mathcal{T}$ è una base perché $\mathcal{T}$ è lin. Ind. per l'esercizio precedente e $|\mathcal{T}| = n+1 = \dim \mathbb{R}[x]_{\leq n}$.

Esercizio: Chi è b_k in $(*)$? $\quad b_1 = \frac{d}{dx} p(x)\big|_{x=a}$

$k = 0, \dots, n$.