Matrice identità

Dato V sp. vett. esiste sempre la funzione identità $Id_V : V \to V$ t.c. $Id_V(u) = u \quad \forall u \in V$

Oss: è lineare

Se scelgo una base B di V

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "F_B"'] & V \arrow[d, "F_B"] \\ \mathbb{K}^n \arrow[r, "S_{\mathbb{1}_n}"'] & \mathbb{K}^n \end{tikzcd}$$

$\mathbb{1}_n$ è la matrice associata a Id_V si chiama matrice identità

$$\mathbb{1}_n = \left(e_1 \middle| e_2 \middle| \dots \middle| e_n \right)$$

$$\mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Oss: $\mathbb{1}_n$ è diagonale

(Una matrice si dice diagonale se può avere coeff. $\neq 0$ solo sulla diagonale)

Oss: i) $\mathbb{1}_n \cdot A = A \quad \forall A \in Mat_{n \times m}(\mathbb{K})$ $(\forall m)$

ii) $A \cdot \mathbb{1}_n = A \quad \forall A \in Mat_{m \times n}(\mathbb{K})$ $(\forall m)$

i)

$$\begin{tikzcd}
V \arrow[r, "L"] \arrow[d] & W \arrow[r, equal, "Id_W"] \arrow[d] & W \arrow[d] \\
\mathbb{K}^m \arrow[r, "S_A"'] \arrow[rr, bend right, "S_{\mathbb{1}_n \cdot A}"'] & \mathbb{K}^n \arrow[r, "S_{\mathbb{1}_n}"'] & \mathbb{K}^n
\end{tikzcd}$$

$$L = Id \circ L$$

$$\Updownarrow$$

$$S_A = S_{\mathbb{1}_n \cdot A}$$

Richiamo sui cambi di base e sulla similitudine

Def: $L_1 : V_1 \to W_1$

mappe lineari

$L_2 : V_2 \to W_2$

$L_1 \sim L_2$ quando

"simile"

$$\begin{tikzcd} V_1 \arrow[r, "L_1"] \arrow[d, "F_V"'] & W_1 \arrow[d, "F_W"] \\ V_2 \arrow[r, "L_2"'] & W_2 \end{tikzcd}$$

$\exists\, F_V, F_W$ invertibili

t.c. $L_2 \circ F_V = F_W \circ L_1$

Oss: $F_W^{-1} \circ L_2 \circ F_V = L_1$

Obiettivo: vogliamo capire quando $L_1 \sim L_2$

Supponiamo che $L_1 \sim L_2$

Oss 1: $\dim V_1 = \dim V_2$ (perché F_V è invertibile)

$\dim W_1 = \dim W_2$ (" F_W ")

Oss 2: $rk L_1 = rk L_2$

$$\begin{tikzcd} V_1 \arrow[r, "L_1"] \arrow[d, "F_V"'] & W_1 \arrow[d, "F_W", "\wr"'] & \supseteq \mathrm{Im}(L_1) \arrow[d, phantom, "\wr"] \\ V_2 \arrow[r, "L_2"'] & W_2 & \supseteq \mathrm{Im}(L_2) \end{tikzcd}$$

$u \in V_1$

$L_1(u) \mapsto F_W(L_1(u)) = L_2(F_V(u))$

questo significa che $\dim \mathrm{Im} L_1 = \dim \mathrm{Im} L_2$

Teorema:

$$L_1 \sim L_2 \iff \begin{array}{l} \dim V_1 = \dim V_2 \quad e \\ \dim W_1 = \dim W_2 \quad e \\ \operatorname{rk} L_1 = \operatorname{rk} L_2 \end{array}$$

Domanda: con quali basi è meglio esprimere la matrice associata a $L: V \to W$?

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "F_{B_V}"'] & W \arrow[d, "F_{B_W}"] \\ \mathbb{K}^n \arrow[r, "S_A"'] & \mathbb{K}^m \end{tikzcd}$$

Es: (Es 2 sett 4)

$L: V \to W$

$\mathcal{B}_1 = \{v_1, v_2, v_3\}$

$\mathcal{B}_2 = \{w_1, w_2, w_3, w_4\}$

con queste basi la mat. assoc. è

$$\begin{pmatrix} 2 & 1 & 3 \\ 3 & 0 & 6 \\ -1 & -1 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

$$\ker L = \langle -2v_1 + v_2 + v_3 \rangle$$

$$\to \mathcal{B}'_V = (\underbrace{-2v_1 + v_2 + v_3}_{u},\ v_1, v_2)$$

$$\mathcal{B}_{\mathrm{Im}L} = (L(v_1), L(v_2))$$

$$\to \mathcal{B}'_W = (L(v_1), L(v_2), w_1, w_2)$$

$$\begin{array}{ccc} V & \xrightarrow{L} & W \\ \downarrow F_{\mathcal{B}'_V} & & \downarrow F_{\mathcal{B}'_W} \\ \mathbb{K}^3 & \longrightarrow & \mathbb{K}^4 \end{array}$$

$$\left(F(L(u)) \,\middle|\, F(L(v_1)) \,\middle|\, F(L(v_2)) \right)$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In generale costruiamo basi in questo modo:

$$L : V \to W$$

$$\mathcal{B}_{\ker L} = (v_1, \dots, v_{\dim \ker L})$$

$$\mathcal{B}_V = (v_{\dim \ker L + 1}, \dots, v_{\dim V}, v_1, \dots, v_{\dim \ker L})$$

$$\mathcal{B}_{\mathrm{Im} L} = (L(\,''\,), \dots, L(\,''\,))$$

$$\mathcal{B}_W = (\underbrace{L(\,''\,), \dots, L(\,''\,)}_{r}, w_{r+1}, \dots, w_{\dim W})$$

$r = \mathrm{rk}\, L$

la matrice associata a L è $\left(e_1 \,\middle|\, e_2 \,\middle|\, e_3 \,\middle|\, \dots \,\middle|\, e_r \,\middle|\, 0 \,\middle|\, \dots \,\middle|\, 0 \right)$

Es di prima:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right) = \left(\begin{array}{c|c} \mathbb{1}_2 & O_{2\times 1} \\ \hline O_{2\times 2} & O_{1\times 1} \end{array}\right)$$

scrittura "a blocchi"

In generale

$$A = \left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$\underbrace{e_1 | \dots | e_r}\quad \underbrace{0 | \dots | 0}$$

Teorema:

$$L_1 \sim L_2 \iff \begin{array}{l} \dim V_1 = \dim V_2 = n \\ \quad e \\ \dim W_1 = \dim W_2 = m \\ \quad e \\ \operatorname{rk} L_1 = \operatorname{rk} L_2 = r \end{array}$$

$\Leftarrow$

Dim:

$$\begin{tikzcd}
V_1 \arrow[r, "L_1"] \arrow[d, "F_{B_{V_1}}"'] & W_1 \arrow[d, "F_{B_{W_1}}"] \\
\mathbb{K}^n \arrow[r, "S_A"] \arrow[d, equal] & \mathbb{K}^m \\
\mathbb{K}^n \arrow[r, "S_B"] & \mathbb{K}^m \\
V_2 \arrow[r, "L_2"] \arrow[u, "F_{B_{V_2}}"] & W_2 \arrow[u, "F_{B_{W_2}}"']
\end{tikzcd}$$

sto applicando 2 volte la procedura di prima

$$A = \left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$\|$$

$$B = \left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & 0 \end{array}\right)$$

$$
\begin{tikzcd}
V_1 \arrow[r, "L_1"] \arrow[d, "F_{B_{V_1}}"'] & W_1 \arrow[d, "F_{B_{W_1}}"] \\
\mathbb{K}^n \arrow[r, "S_A"] & \mathbb{K}^n \\
V_2 \arrow[r, "L_2"] \arrow[u, "F_{B_{V_2}}"] & W_2 \arrow[u, "F_{B_{W_2}}"']
\end{tikzcd}
$$

quindi:

$$
\begin{tikzcd}
V_1 \arrow[r, "L_1"] \arrow[d, "F_{B_{V_2}}^{-1} \circ F_{B_{V_1}}"'] & W_1 \arrow[d, "F_{B_{W_2}}^{-1} \circ F_{B_{W_1}}"] \\
V_2 \arrow[r, "L_2"'] & W_2
\end{tikzcd}
\Rightarrow L_1 \sim L_2
$$

Esercizi sulla moltiplicazione di matrici

Es 4 Sett 4 : $L : \mathbb{K}^4 \to \mathbb{K}^4$

$$e_1 + e_2 \mapsto e_2$$
$$e_1 - e_2 \mapsto e_1$$
$$e_3 + e_4 \mapsto e_3$$
$$e_3 - e_4 \mapsto e_4$$

$L = S_A$

$$A = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & -1/2 \end{pmatrix} = \left(\begin{array}{cc|cc} 1/2 & -1/2 & \multicolumn{2}{c}{0} \\ 1/2 & 1/2 & & \\ \hline \multicolumn{2}{c|}{0} & 1/2 & 1/2 \\ & & 1/2 & -1/2 \end{array}\right)$$

$$B = A^{-1} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix}$$

f invertibile $\Longleftrightarrow \exists g : \begin{array}{l} fg = id \\ gf = id \end{array}$

- - - - - - - - - - - -

L è invertibile $\Rightarrow \exists L^{-1}$

$L \downarrow A$ $\qquad L^{-1} \downarrow B$ $\qquad \downarrow ?$

$AB = \mathbb{1}_4$

verifichiamo queste identità : $BA = \mathbb{1}_4$

$$\begin{pmatrix} \boxed{\begin{matrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{matrix}} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} & \begin{matrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{matrix} \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} = \begin{pmatrix} A_1 B_1 + 0 \cdot 0 & A_1 \cdot 0 + 0 \cdot B_2 \\ 0 \cdot B_1 + A_2 \cdot 0 & 0 \cdot 0 + A_2 B_2 \end{pmatrix} =$$

$$= \begin{pmatrix} A_1 B_1 & 0 \\ 0 & A_2 B_2 \end{pmatrix}$$

$$A_1 B_1 = \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1}_2$$

$$A_2 B_2 = \qquad\qquad = \mathbb{1}_2$$

$$= \begin{pmatrix} \mathbb{1}_2 & 0 \\ 0 & \mathbb{1}_2 \end{pmatrix} = \mathbb{1}_4$$

L'importante è che i blocchi abbiano taglie compatibili:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^2 = \left(\begin{array}{c|cc} 1 & 1 & 1 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right)^2 = \left(\begin{array}{c|c} \mathbb{1}_1 & A \\ \hline 0 & \mathbb{1}_2 \end{array}\right)^2$$

$$\begin{pmatrix} \mathbb{1}_1 & A \\ 0 & \mathbb{1}_2 \end{pmatrix}\begin{pmatrix} \mathbb{1}_1 & A \\ 0 & \mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} \mathbb{1}_1\mathbb{1}_1 + A \cdot 0 & \mathbb{1}_1 \cdot A + A \cdot \mathbb{1}_2 \\ 0\mathbb{1}_1 + \mathbb{1}_2 \cdot 0 & 0 \cdot A + \mathbb{1}_2 \cdot \mathbb{1}_2 \end{pmatrix}$$

$$= \begin{pmatrix} \mathbb{1}_1 & 2A \\ 0 & \mathbb{1}_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \cancel{\mathbb{1}_2 \cdot A}$$

In generale non mi sogno nemmeno di pensare

$$\cancel{AB = BA}$$ con A, B matrici

AB $\quad A \in Mat_{n \times m} \quad B \in Mat_{m \times p} \rightarrow n \times p$

BA solo se $n = p \rightarrow m \times m$

se $n \neq m$ $\quad AB, BA$ non hanno la stessa taglia

$n = m = p$

Es: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad AB = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \neq BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Se $AB = BA$ dico che A, B "commutano"

Es $\qquad \mathbb{K}^3 \xrightarrow{L_1} \mathbb{K}^2 \qquad \mathbb{K}^2 \xrightarrow{L_2} \mathbb{K}^2$

$$L_1\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}1\\2\end{pmatrix} \qquad L_2\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

$$L_1\begin{pmatrix}0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} \qquad L_2\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}1\\1\end{pmatrix}$$

$$L_1\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}1\\4\end{pmatrix}$$

Trovare la matrice associata M a $L_2 \circ L_1$ nelle basi standard di $\mathbb{K}^3$ e $\mathbb{K}^2$.

Sol: L_1 in basi $B_1 = \left(\begin{pmatrix}1\\1\\0\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\0\\1\end{pmatrix}\right)$ $\quad B_2 =$ standard

$$L_1 \longleftrightarrow A = \begin{pmatrix}1 & 1 & 1\\ 2 & 3 & 4\end{pmatrix}$$

$$L_2 \longleftrightarrow \qquad B = \begin{pmatrix}3 & 1\\ 2 & 1\end{pmatrix}$$

B_3, B_4 analog.

- $$\begin{tikzcd}
\mathbb{K}^3 \arrow[r, equal] \arrow[d, "F"'] \arrow[rrrr, bend left, "A_2"] & \mathbb{K}^3 \arrow[r, "L_1"] \arrow[d, "F_{B_1}"'] & \mathbb{K}^2 \arrow[r, equal] \arrow[d, "F"'] & \mathbb{K}^2 \arrow[r, "L_2"] \arrow[d, "F_{B_3}"'] & \mathbb{K}^2 \arrow[d, "F"] \\
\mathbb{K}^3 & \mathbb{K}^3 \arrow[l, "C_{B_1,std}"] \arrow[r, "S_A"'] & \mathbb{K}^2 & \mathbb{K}^2 \arrow[l, "C_{B_3,std}"] \arrow[r, "S_B"'] & \mathbb{K}^2
\end{tikzcd}$$

- $\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$

- moltiplicare:

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \end{pmatrix} \cdot \underbrace{\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}^{-1}}_{= \begin{pmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{pmatrix}} \cdot \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix} = M$$