1. Comunicazioni.
2. Domande?
3. Tecniche di calcolo:
   - base del nucleo
   - base dell'immagine
   - inversa

di applica~~zioni~~ lineari matrici.

Algoritmo di eliminazione di Gauss

Per illustrarlo facciamo un esempio:

Es:

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \\ 0 & 1 & 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \end{pmatrix} \in Mat_{4\times 9}(\mathbb{R})$$

$$S_A : \mathbb{R}^9 \longrightarrow \mathbb{R}^4$$

Vogliamo calcolare una base di Ker A.

$$\mathcal{B}_{Col(A)} = \left\{ A^2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, A^5 = \begin{pmatrix} 1 \\ 2 \\ -1 \\ 2 \end{pmatrix}, A^7 = \begin{pmatrix} 2 \\ 3 \\ 1 \\ 3 \end{pmatrix} \right\}$$

Estendiamola ad una base di $\mathbb{R}^4$ $\mathcal{B} = \{A^2, A^5, A^7, e_1\}$.

$$\begin{array}{ccc} \mathbb{R}^9 & \xrightarrow{\quad A \quad} & \mathbb{R}^4 \\ Id = F_e \ \Vert & & \downarrow F_{\mathcal{B}} = S_C \\ \mathbb{R}^9 & \longrightarrow & \mathbb{R}^4 \end{array}$$

$$R = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$
\begin{tikzcd}
\mathbb{R}^9 \arrow[r, "A"] \arrow[d, equal, "\mathrm{Id} = F_e"'] & \mathbb{R}^4 \arrow[d, "F_{\mathcal{B}} = S_C"] \\
\mathbb{R}^9 \arrow[r] & \mathbb{R}^4
\end{tikzcd}
$$

$$
R = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

$$\operatorname{Ker} A = \operatorname{Ker} R =$$

$$
= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_9 \end{pmatrix} \in \mathbb{R}^9 \;\middle|\; \begin{array}{l} x_2 + 2x_3 + 3x_4 + x_6 - x_8 + x_9 = 0, \\ x_5 + x_6 + 2x_9 = 0, \\ x_7 + x_8 + x_9 = 0. \end{array} \right\}
$$

$$
= \left\{ X \;\middle|\; \begin{array}{l} x_2 = -2x_3 - 3x_4 - x_6 + x_8 - x_9 \\ x_5 = -x_6 - 2x_9 \\ x_7 = -x_8 - x_9 \end{array} \right\}
$$

$$\operatorname{Ker} A = \operatorname{Ker} R =$$

$$= \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_9 \end{pmatrix} \in \mathbb{R}^9 \;\middle|\; \begin{array}{r} x_2 + 2x_3 + 3x_4 + x_6 - x_8 + x_9 = 0, \\ x_5 + x_6 \qquad + 2x_9 = 0, \\ x_7 + x_8 + x_9 = 0. \end{array} \right\}$$

$$= \left\{ X \;\middle|\; \begin{array}{rl} x_2 = & -2x_3 - 3x_4 - x_6 + x_8 - x_9 \\ x_5 = & \qquad\qquad -x_6 \quad -2x_9 \\ x_7 = & \qquad\qquad\qquad -x_8 - x_9 \end{array} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ -2x_3 - 3x_4 - x_6 + x_8 - x_9 \\ x_3 \\ x_4 \\ -x_6 - 2x_9 \\ x_6 \\ -x_8 - x_9 \\ x_8 \\ x_9 \end{pmatrix} \;\middle|\; x_1, x_3, x_4, x_6, x_8, x_9 \in \mathbb{R} \right\}$$

$$= \left\{ X \;\middle|\; \begin{aligned} x_2 &= -2x_3 - 3x_4 - x_6 + x_8 - x_9 \\ x_5 &= -x_6 - 2x_9 \\ x_7 &= -x_8 - x_9 \end{aligned} \right\}$$

$$= \left\{ \begin{pmatrix} x_1 \\ -2x_3 - 3x_4 - x_6 + x_8 - x_9 \\ x_3 \\ x_4 \\ -x_6 - 2x_9 \\ x_6 \\ -x_8 - x_9 \\ x_8 \\ x_9 \end{pmatrix} \;\middle|\; x_1, x_3, x_4, x_6, x_8, x_9 \in \mathbb{R} \right\}$$

$$= \left\langle \underset{\text{"}x_1\text{"}=1}{\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{\text{"}x_3=1\text{"}}{\begin{pmatrix} 0 \\ -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{\text{"}x_4=1\text{"}}{\begin{pmatrix} 0 \\ -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{\text{"}x_6=1\text{"}}{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}}, \underset{\text{"}x_8=1\text{"}}{\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \\ 1 \\ 0 \end{pmatrix}}, \underset{\text{"}x_9=1\text{"}}{\begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}} \right\rangle$$

R si chiama la forma di Hérmite oppure la forma a scala ridotta di A.
("Row Reduced Echelon Form" di A)

In MATLAB:

$$R = \mathrm{rref}(A)$$

Idea: Data A trovare una matrice R t.c.

1) $\operatorname{Ker} A = \operatorname{Ker} R$

2) $\operatorname{Ker} R$ è facile da calcolare.
(?)

Matrici generiche in MATLAB :

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: ① syms a, b, c, d

$A = \text{sym}([a, b; c, d])$

② $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ A= sym ('a', [m, n])

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \\ 0 & 1 & 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \end{pmatrix}$$

$$\begin{array}{l} R = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{array}{ccccccccc} \quad & e_1 & & & e_2 & & e_3 & & \\ & x_2 & & & x_5 & & x_7 & & \end{array} \end{array}$$

$x_2 = \ldots$ $\nexists\, x_5, x_7$

$x_5 = \ldots$ $\nexists\, x_2, x_7$

$x_7 = \ldots$ $\nexists\, x_2, x_5$

$$R = \begin{pmatrix} 0 & 1 & 2 & 3 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

a scala

$$
\begin{tikzcd}
\mathbb{R}^9 \arrow[r, "A"] \arrow[d, equal] & \mathbb{R}^4 \arrow[d, "F_{\mathcal{B}} = S_C"] \\
\mathbb{R}^9 \arrow[r, "R"] & \mathbb{R}^4
\end{tikzcd}
$$

$$\boxed{CA = R}$$

C è invertibile.

$$\mathcal{B} = \{ A^2, A^5, A^7, e_1 \}$$

$$C = \left(A^2 | A^5 | A^7 | e_1 \right)^{-1}$$

Vogliamo R : la matrice che rappresenta A nella base canonica in partenza e nella base di arrivo :

$$\mathcal{B} = \{ A^{j_1}, \dots, A^{j_r}, e_{i_{r+1}}, \dots, e_{i_m} \}$$

$\Leftrightarrow$ Trovare C invertibile t.c. $CA = R$

operazioni elementari sulle righe di una matrice

(I) Scambio di due righe:

"$R_i \longleftrightarrow R_j$" "$A_i \longleftrightarrow A_j$"

(II) Moltiplico una riga per uno scalare non-nullo:

$$R_i \longmapsto \lambda R_i \quad \text{per } \lambda \neq 0$$

(III) Aggiungere alla riga i-esima un multiplo della riga j-esima

$$R_i \longmapsto R_i + k R_j \quad \text{per qualche } k.$$

$$A = \begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \\ 0 & 1 & 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \end{pmatrix}$$

Vogliamo trasformare A in R usando le tre operazioni elementari:

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \\ 0 & 1 & 2 & 3 & -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 & 2 & 3 & 3 & 2 & 8 \end{pmatrix} \xrightarrow[\substack{R_2 \mapsto R_2 - R_1 \\ R_3 \mapsto R_3 - R_1 \\ R_4 \mapsto R_4 - R_1}]{} \begin{array}{c} \quad e_1 \qquad\qquad\qquad\qquad\qquad\qquad \\ \begin{pmatrix} 0 & \boxed{1} & 2 & 3 & 1 & 2 & 2 & 1 & 5 \\ 0 & \boxed{0} & 0 & 0 & 1 & 1 & 1 & 1 & 3 \\ 0 & \boxed{0} & 0 & 0 & -2 & -2 & -1 & -1 & -5 \\ 0 & \boxed{0} & 0 & 0 & 1 & 1 & 1 & 1 & 3 \end{pmatrix} \end{array}$$

$$\xrightarrow[\substack{R_1 \mapsto R_1 - R_2 \\ R_3 \mapsto R_3 + 2R_2 \\ R_4 \mapsto R_4 - R_2}]{} \begin{array}{c} \begin{pmatrix} 0 & \boxed{1} & 2 & 3 & \boxed{0} & 1 & 1 & 0 & 2 \\ 0 & \boxed{0} & 0 & 0 & \boxed{1} & 1 & 1 & 1 & 3 \\ 0 & \boxed{0} & 0 & 0 & \boxed{0} & 0 & 1 & 1 & 1 \\ 0 & \boxed{0} & 0 & 0 & \boxed{0} & 0 & 0 & 0 & 0 \end{pmatrix} \\ \qquad e_1 \qquad\qquad\quad e_2 \qquad\qquad\qquad \end{array}$$

$$
\begin{array}{c}
\left(\begin{array}{ccccccccc}
0 & \boxed{1} & 2 & 3 & \boxed{0} & 1 & 1 & 0 & 2 \\
0 & \boxed{0} & 0 & 0 & \boxed{1} & 1 & 1 & 1 & 3 \\
0 & \boxed{0} & 0 & 0 & \boxed{0} & 0 & 1 & 1 & 1 \\
0 & \boxed{0} & 0 & 0 & \boxed{0} & 0 & 0 & 0 & 0
\end{array}\right) \\
\begin{array}{ccccccccc}
 & e_1 & & & e_2 & & & &
\end{array}
\end{array}
\xrightarrow[R_2 \mapsto R_2 - R_3]{R_1 \mapsto R_1 - R_3}
\begin{array}{c}
\left(\begin{array}{ccccccccc}
0 & \boxed{1} & 2 & 3 & \boxed{0} & 1 & \boxed{0} & -1 & 1 \\
0 & \boxed{0} & 0 & 0 & \boxed{1} & 1 & \boxed{0} & 0 & 2 \\
0 & \boxed{0} & 0 & 0 & \boxed{0} & 0 & \boxed{1} & 1 & 1 \\
0 & \boxed{0} & 0 & 0 & \boxed{0} & 0 & \boxed{0} & 0 & 0
\end{array}\right) \\
\shortparallel \\
R
\end{array}
$$

Es:

$$A = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 3 & 1 & 4 \\ 0 & 2 & 0 & 2 \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\rightsquigarrow} \begin{pmatrix} 0 & 3 & 1 & 4 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$\underset{R_1 \mapsto \frac{1}{3} R_1}{\rightsquigarrow} \begin{pmatrix} 0 & 1 & 1/3 & 4/3 \\ 0 & 0 & 1 & 1 \\ 0 & 2 & 0 & 2 \end{pmatrix}$$

$$\underset{R_3 \mapsto R_3 - 2R_1}{\rightsquigarrow} \begin{pmatrix} 0 & \boxed{1} & 1/3 & 4/3 \\ 0 & \boxed{0} & 1 & 1 \\ 0 & \boxed{0} & -2/3 & -2/3 \end{pmatrix}$$

$$2 - 2\left(\frac{4}{3}\right) = 2 - \frac{8}{3} = \frac{6-8}{3} = -\frac{2}{3}$$

$$\underset{\substack{R_1 \mapsto R_1 - \frac{1}{3} R_2 \\ R_3 \mapsto R_3 + \frac{2}{3} R_2}}{\rightsquigarrow} \begin{array}{c} \begin{pmatrix} 0 & \boxed{1} & \boxed{0} & 1 \\ 0 & \boxed{0} & \boxed{1} & 1 \\ 0 & \boxed{0} & \boxed{0} & 0 \end{pmatrix} \\ \quad e_1 \quad e_2 \end{array} = \mathrm{rref}(A).$$