Richiami: $\det : \mathrm{Mat}_{n\times n}(\mathbb{K}) \to \mathbb{K}$ t.c.

$$\det(P_{ij}A) = -\det(A)$$
$$\det(D_i(\lambda)A) = \lambda \det(A)$$
$$\det(F_{ij}(c)A) = \det(A)$$
$$\det(\mathbb{1}_n) = 1$$

Prop.: $\det(A) \neq 0 \iff \mathrm{rref}(A) = \mathbb{1}_n$
$\iff A$ è invertibile.

$$\det\begin{pmatrix} a_{11} & * & \cdots & * \\ 0 & a_{22} & & \vdots \\ \vdots & & \ddots & * \\ 0 & \cdots & 0 & a_{nn} \end{pmatrix} = a_{11} a_{22} \cdots a_{nn}$$

$$\det^{(n)}(A) = \sum_{j=1}^{n} a_{ij} \underbrace{\det^{(n-1)}(A_{i,j})\,(-1)^{i+j}}_{C_{i,j}}$$

sviluppo di Laplace della riga i.

operazioni elementari sulle colonne:

$$A \xrightarrow{C^i \leftrightarrow C^j} A P_{ij}$$

$$A \xrightarrow{C^i \mapsto \lambda C^i} A D_i(\lambda)$$

$$A \xrightarrow{C^i \mapsto C^i + \lambda C^j} A F_{ji}(\lambda)$$

$$\left.\right\} \quad (AB)^t = B^t A^t$$

$$\det(A) = \sum_{i=1}^{n} a_{ij} \det(A_{i,j}) (-1)^{i+j}$$

Sviluppo di Laplace della colonna j

Teo: $\mathrm{rg}(A^t) = \mathrm{rg}(A)$

$$\det(A^t) = \det(A).$$

Per calcolare det(A)

1) operare sulle righe o sulle colonne di A per creare una riga o una colonna contenente tanti zeri.

2) Sviluppare quella riga o quella colonna.

Utilizzo del determinante per il calcolo dell'inversa

<u>Problema</u> : data $A \in Mat_{n\times n}(\mathbb{K})$ invertibile, calcolare A^{-1}.

Se $n=2$ abbiamo dimostrato:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A}\underbrace{\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}}$$

<u>matrice</u>
<u>Aggiunta</u>
<u>di $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$</u>

Def: Dato $n \geq 1$ ed una matrice qualsiasi $A \in Mat_{n \times n}(\mathbb{K})$ e dati un indice di riga i ed un indice di colonna j, definiamo il <u>cofattore (i,j) di A</u> come il numero

$$C_{i,j} = C_{i,j}(A) = \det(A_{i,j})(-1)^{i+j}$$

<u>OSS</u>: $\det(A) = \sum_{i=1}^{n} a_{ij} C_{ij} = \sum_{j=1}^{n} a_{ij} C_{i,j}$

La <u>matrice aggiunta di A</u> è la matrice $Agg(A)$ che ha come componenti i cofattori:

$$Agg(A) = \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ C_{21} & C_{22} & \cdots & C_{2n} \\ \vdots & & & \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}$$

Es: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$C_{11}(A) = d \qquad C_{12} = -c$

$C_{21} = -b \qquad C_{22} = a$

$$Agg(A) = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{\det A} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det A} Agg(A)^t$$

Teorema: Sia $A \in Mat_{n \times n}(\mathbb{K})$. Allora

$$A \, Agg(A)^t = \det(A) \, \mathbb{1}_n = \begin{pmatrix} \det(A) & 0 & \cdots & 0 \\ 0 & \det(A) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \det(A) \end{pmatrix}$$

Es:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \det(A) \, \mathbb{1}_2 .$$

dim (Teorema): Dobbiamo dimostrare

$$\left[A \, Agg(A)^t \right]_i^j = \begin{cases} \det(A) & \text{se } i = j \\ 0 & \text{altrimenti} \end{cases}$$

$$\left[A \, Agg(A)^t\right]_i^i = A_i \left(Agg(A)^t\right)^i$$

$$= A_i \begin{pmatrix} c_{i1} \\ c_{i2} \\ \vdots \\ c_{in} \end{pmatrix} = a_{i1}c_{i1} + a_{i2}c_{i2} + \dots + a_{in}c_{in}$$

$$= \det(A).$$

Sviluppo di Laplace della i-esima riga.

$$\left[A\,Agg(A)^t\right]_i^j \overset{i\neq j}{=} A_i\,\left(Agg(A)^t\right)^j$$

$$= A_i\,\left(Agg(A)_j\right)^t$$

$$= \sum_{k=1}^{n} a_{ik}\,Agg(A)_j^k$$

$$= \sum_{k=1}^{n} a_{ij}\,c_{jk}$$

Sia B la matrice che ha le stesse righe di A a parte la j-esima che è uguale ad A_i

$$B = \begin{pmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_i \\ \vdots \\ A_n \end{pmatrix} \quad \begin{array}{c} \\ \\ i \\ \\ j \\ \\ \\ \end{array} \quad A = \begin{pmatrix} A_1 \\ \vdots \\ A_i \\ \vdots \\ A_j \\ \vdots \\ A_n \end{pmatrix}$$

$$0 = \det(B) \overset{\text{j-esima riga di } B}{=} \sum_{k=1}^{n} b_{jk}\, c_{jk}(B)$$

$$= \sum_{k=1}^{n} a_{ik}\, c_{jk}(B)$$

$$= \sum_{k=1}^{n} a_{ik}\, c_{jk}(A).$$

$\blacksquare$

<u>COR</u>: Se A è invertibile,

$$A^{-1} = \frac{1}{\det(A)} \operatorname{Agg}(A)^t$$

<u>Formula di Cramer per l'inversa</u>

<u>dim</u>: Per il Teorema

$$A \operatorname{Agg}(A)^T = \det(A)\, \mathbb{1}_n \overset{\det(A)\neq 0}{\Longrightarrow} \frac{1}{\det A} A \operatorname{Agg}(A)^t = \mathbb{1}_n$$

$$\Longrightarrow A^{-1} = \frac{1}{\det(A)} \operatorname{Agg}(A)^t.$$

Es: Calcolare l'inversa di

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 2 & 2 & 1 \\ 1 & -1 & \frac{3}{2} \end{pmatrix}$$

utilizzando la formula di Cramer.

Sol.:

$$\det A = \det \begin{pmatrix} 1 & -1 & 1 \\ 0 & 4 & -1 \\ 0 & 0 & \frac{1}{2} \end{pmatrix} = 2$$

$$C_{11} = \det \begin{pmatrix} 2 & 1 \\ -1 & 3/2 \end{pmatrix} = 4, \quad C_{21} = -\det \begin{pmatrix} -1 & 1 \\ -1 & 3/2 \end{pmatrix} = \frac{1}{2}, \quad C_{31} = \det \begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix} = -3$$

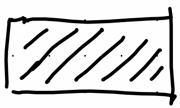
$$C_{12} = -\det \begin{pmatrix} 2 & 1 \\ 1 & 3/2 \end{pmatrix} = -2, \quad C_{22} = \det \begin{pmatrix} 1 & 1 \\ 1 & 3/2 \end{pmatrix} = \frac{1}{2}, \quad C_{32} = -\det \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} = 1$$

$$C_{13} = \det \begin{pmatrix} 2 & 2 \\ 1 & -1 \end{pmatrix} = -4, \quad C_{23} = -\det \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} = 0, \quad C_{33} = \det \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix} = 4.$$

$$\Rightarrow \quad A^{-1} = \frac{1}{2} \begin{pmatrix} 4 & 1/2 & -3 \\ -2 & 1/2 & 1 \\ -4 & 0 & 4 \end{pmatrix}$$

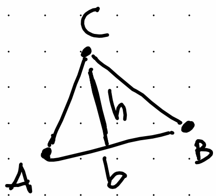
Applicazioni del determinante (continua)

Aree



$A = b \cdot h$

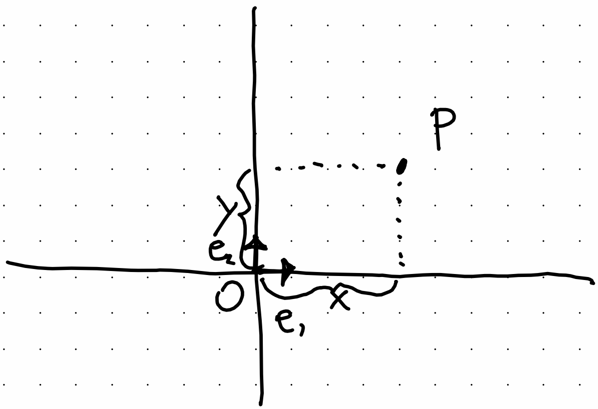
Triangolo



formula per l'area dati 3 punti che sono i vertici del triangolo

$$A_{\text{triangolo}} = \text{Area}(\overset{\triangle}{ABC}) = \frac{b \cdot h}{2}$$

Lavoriamo sul piano cartesiano è simile a V_O^2



$$P = \begin{pmatrix} x \\ y \end{pmatrix}$$

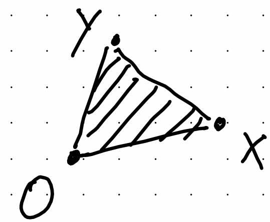
$$V_O^2 \longleftrightarrow \mathbb{R}^2$$

$$\xrightarrow[F_{(e_1, e_2)}]{}$$

$\text{Area}(\hat{ABC})$ dati: $A, B, C \in \mathbb{R}^2 \longleftrightarrow$ piano cartesiano

ottenimo l'area di $\hat{ABC}$ che è un reale (≥ 0)

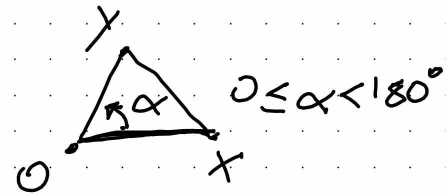
A meno di traslazione assumiamo che $A = O$

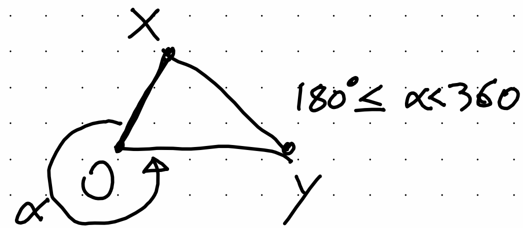
$\text{Area}(X, Y)$ 

$\text{Area}: \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

Area orientata di $\overset{\triangle}{OXY}$

$$A(\overset{\triangle}{OXY}) := \begin{cases} +\,\text{Area}(\overset{\triangle}{OXY}) \\ -\,\text{Area}(\overset{\triangle}{OXY}) \end{cases}$$

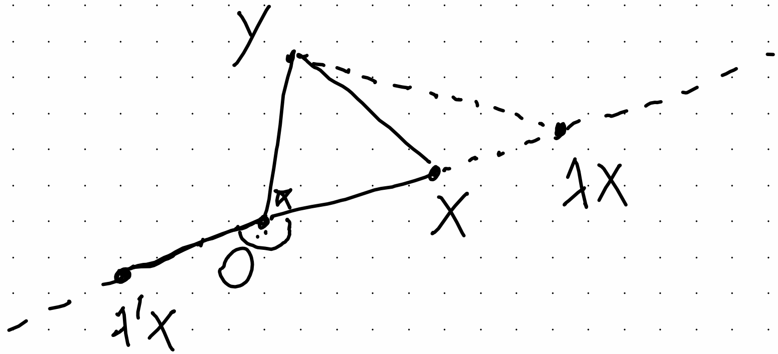




Oss su $A(X,Y) = A(\overset{\triangle}{OXY})$

- $A(x,y) = -A(y,x) \quad \forall\, x,y$

$\boxed{}$ $A(\lambda x, y) = \lambda A(x, y)$

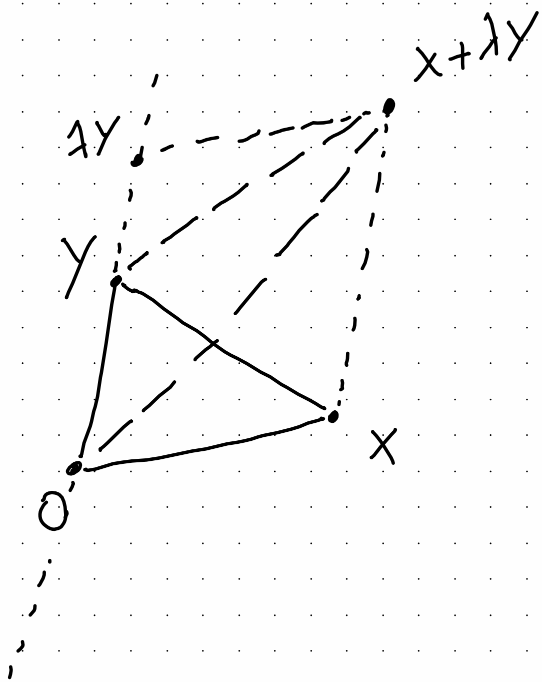


- se $\lambda \geq 0$ $\quad \lambda \operatorname{Area}(x, y) = \operatorname{Area}(\lambda x, y)$

- se $\lambda' < 0$ $\quad (-\lambda')\operatorname{Area}(x, y) = \operatorname{Area}(\lambda' x, y)$

sto cambiando segno all'area orientata

$\blacksquare$ $$A(x+\lambda y, y) = A(x, y)$$

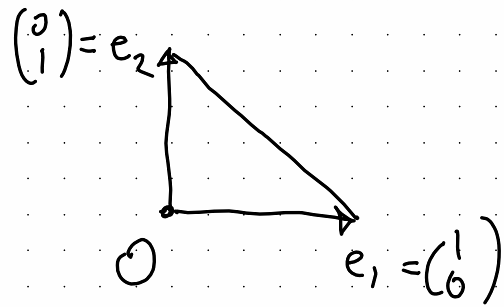


Riassumendo

- $A(X,Y) = -A(Y,X)$
- $A(\lambda X,Y) = \lambda A(X,Y)$
- $A(X+\lambda Y, Y) = A(X,Y)$

$A(e_1,e_2) \neq 1$?

- $A(e_1,e_2) = \frac{1}{2}$



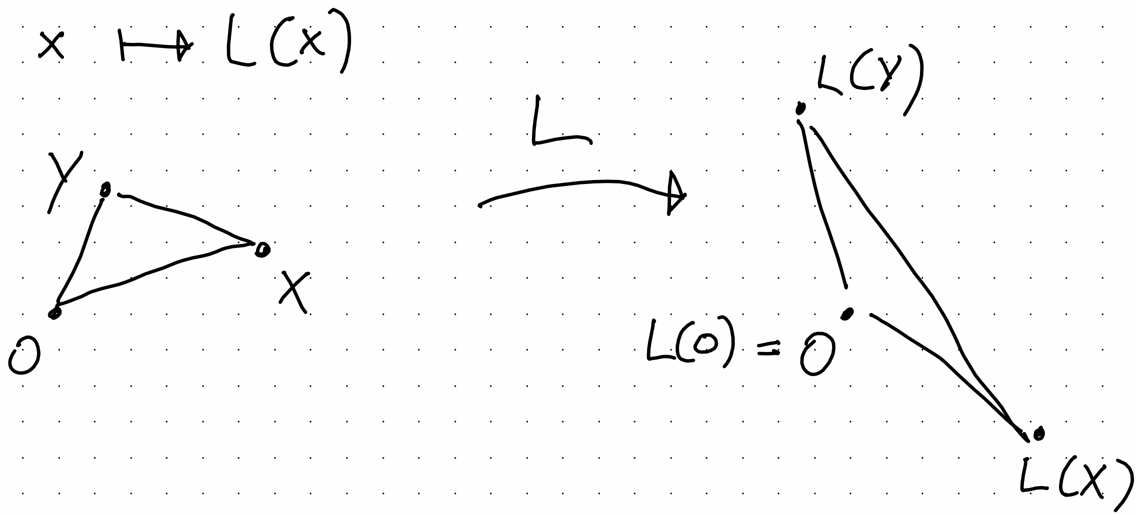
$$2A(X,Y) = \det(X|Y)$$

$$\text{Area}(XY) = \left|\frac{1}{2}\det(X|Y)\right|$$

$L : \mathbb{R}^2 \to \mathbb{R}^2$ lineare

$L = S_B$

$x \mapsto L(x)$



$$A(L(X), L(Y)) = \frac{1}{2}\det(L(X) \mid L(Y))$$
$$= \frac{1}{2}\det(BX \mid BY) = \frac{1}{2}\det(B\cdot(X\mid Y))$$
$$= \det B \cdot \frac{1}{2}\det(X\mid Y) = \det B \cdot A(X,Y)$$

Matrice di Vandermonde

$x_1, \ldots, x_n \in \mathbb{K}$

$$V(x_1, \ldots, x_n) = \begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & \ddots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix}$$

$$V(0,1,2) = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}$$

L'abbiamo già vista:

$$V = \mathbb{K}[x]_{\leq n-1} \qquad F(p(x)) = \begin{pmatrix} p(x_1) \\ p(x_2) \\ \vdots \\ p(x_n) \end{pmatrix}$$

$$\begin{tikzcd} V \arrow[r, equal] \arrow[d, "F_{\mathcal{E}}"'] & V \arrow[d, "F"] \\ \mathbb{K}^n \arrow[r, dashed, "S_C"'] & \mathbb{K}^n \end{tikzcd}$$

Oss: $C = V(x_1, \dots, x_n)$

$$C^1 = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \quad C^2 = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \cdots \quad C^{n-1} = \begin{pmatrix} x_1^{n-1} \\ \vdots \\ x_n^{n-1} \end{pmatrix}$$

S_C è invertibile $\Longleftrightarrow$ $\det C = \det V(x_1, \dots, x_n) \neq 0$

$\Updownarrow$

F è invertibile

$\Updownarrow$

i polinomi di grado $\leq n-1$
sono univocamente e sempre determinati
dalla valutazione sui punti $x_1, \dots, x_n$

$$\det(V(x_1,\dots,x_n)) = \det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & & & \ddots & \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{pmatrix} =$$

$$\det\begin{pmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 0 & x_2-x_1 & x_2^2-x_1^2 & & x_2^{n-1}-x_1^{n-1} \\ 0 & x_3-x_1 & \vdots & \ddots & \\ \vdots & & & & \\ 0 & x_n-x_1 & x_n^2-x_1^2 & & x_n^{n-1}-x_1^{n-1} \end{pmatrix} =$$

$$\det\begin{pmatrix} x_2-x_1 & x_2^2-x_1^2 & & x_2^{n-1}-x_1^{n-1} \\ x_3-x_1 & \vdots & \ddots & \\ & & & \\ x_n-x_1 & x_n^2-x_1^2 & & x_n^{n-1}-x_1^{n-1} \end{pmatrix} =$$

$$= (x_2-x_1)(x_3-x_1)\cdot\ldots\cdot(x_n-x_1)\cdot\det\begin{pmatrix} 1 & \cdots & \frac{x_2^{n-1}-x_1^{n-1}}{x_2-x_1} \\ \vdots & & \\ 1 & \cdots & \frac{x_n^{n-1}-x_1^{n-1}}{x_n-x_1} \end{pmatrix} =$$

opero sulle colonne

$$= (x_2-x_1)(x_3-x_1)\cdot\ldots\cdot(x_n-x_1)\det V(x_2, x_3, \ldots, x_n)$$

$$\vdots$$

$$= \prod_{j>i}(x_j-x_i)\underbrace{\det V(x_n)}_{=1} = \prod_{j>i}(x_j-x_i)$$

$$V(x_n) = (1)$$