Retta per 2 punti di $\mathbb{R}^2$

$P_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \quad P_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \in \mathbb{R}^2 \qquad P_0 \neq P_1 .$

La retta per P_0 e P_1 ha eq. cartesiana

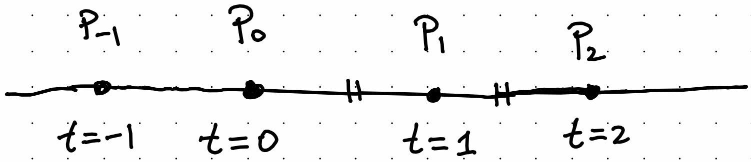
$$r: \quad -(y_1 - y_0)(x - x_0) + (x_1 - x_0)(y - y_0) = 0$$

ed eq. parametrica

$$r = P_0 + \langle v \rangle \ni P_1$$

$\exists\, t \in \mathbb{R}$ t.c. $P_1 = P_0 + tv$. Poiché $t \neq 0$, $P_1 - P_0 = tv$ e possiamo scegliere come vettore direttore di r proprio $P_1 - P_0$

$$r = P_0 + \langle P_1 - P_0 \rangle$$

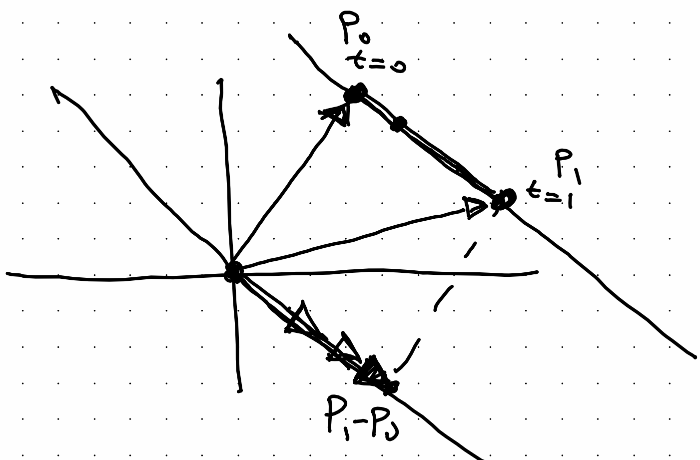


$$r = P_0 + \langle P_1 - P_0 \rangle \ni P_t$$

$$P_t = P_0 + t\,(P_1 - P_0)$$

$$P_t = (1-t)\,P_0 + t\,P_1$$

$$r = \{ t_0 P_0 + t_1 P_1 \mid t_0 + t_1 = 1 \} = \{\text{combinazioni affini di } P_0 \text{ e } P_1\}.$$



$$\overline{P_0 P_1} = \{ P_0 + t\,(P_1 - P_0) \mid 0 \le t \le 1 \}$$

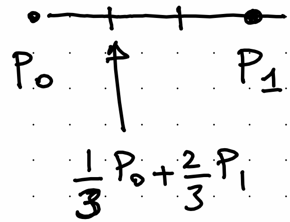
$$= \{ (1-t)\,P_0 + t\,P_1 \mid 0 \le t \le 1 \}$$

$$r \quad = \left\{ t_0 P_0 + t_1 P_1 \;\middle|\; \begin{array}{l} t_0 + t_1 = 1 \\ 0 \le t_0, t_1 \le 1 \end{array} \right\}$$

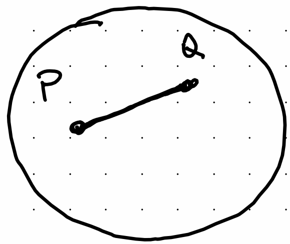
combinazione convessa di P_0 e P_1.

$$r_0 = \langle P_1 - P_0 \rangle$$

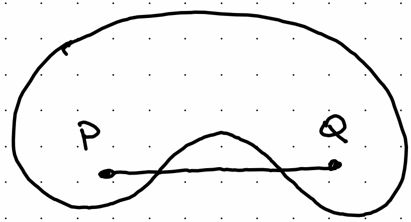
$$\frac{1}{3}P_0 + \frac{2}{3}P_1$$



Un sottoinsieme <u>convesso</u> C di $\mathbb{R}^2$ è un ~~sun~~ sottoinsieme t.c. $\forall\, P, Q \in C$ $\overline{PQ} = \{ t_0 P + t_1 Q \mid t_0 + t_1 = 1,\ t_0, t_1 \geq 0 \} \subset C$



Convesso.



non convesso

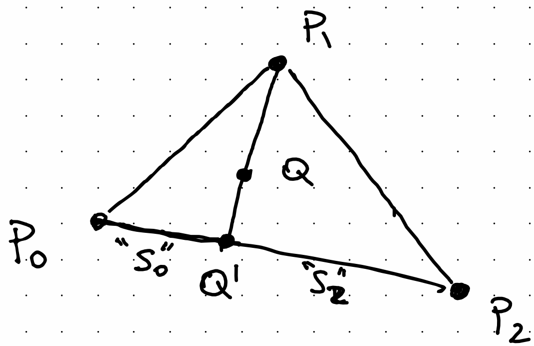
Def: Dati $P_0, P_1, \dots, P_n \in \mathbb{R}^2$, una loro <u>combinazione convessa</u> è un vettore di $\mathbb{R}^2$ della forma

$$v = t_0 P_0 + t_1 P_1 + \dots + t_n P_n$$

t.c. 1) $t_0 + t_1 + \dots + t_n = 1$

2) $t_0, t_1, \dots, t_n \geq 0$

Es:



$Q' = s_0 P_0 + s_2 P_2$ con $s_0 + s_2 = 1$, $s_0, s_2 \geq 0$.

$Q = t_1 P_1 + t' Q'$ con $t_1 + t' = 1$, $t_1, t' \geq 0$

$$Q = t_1 P_1 + t'(s_0 P_0 + s_2 P_2)$$
$$= t_1 P_1 + t' s_0 P_0 + t' s_2 P_2$$

$t_1 + t' s_0 + t' s_2 = 1$

$t_1 \geq 0$, $t' s_0 \geq 0$, $t' s_2 \geq 0$

$$\text{Triangolo}(P_0, P_1, P_2) = \{ t_0 P_0 + t_1 P_1 + t_2 P_2 \mid t_0, t_1, t_2 \geq 0, t_0 + t_1 + t_2 = 1 \}$$

$$\mathrm{Conv}(P_0, P_1, \dots, P_n) = \{ t_0 P_0 + \dots + t_n P_n \mid t_0, \dots, t_n \geq 0,\ t_0 + \dots + t_n = 1 \}$$

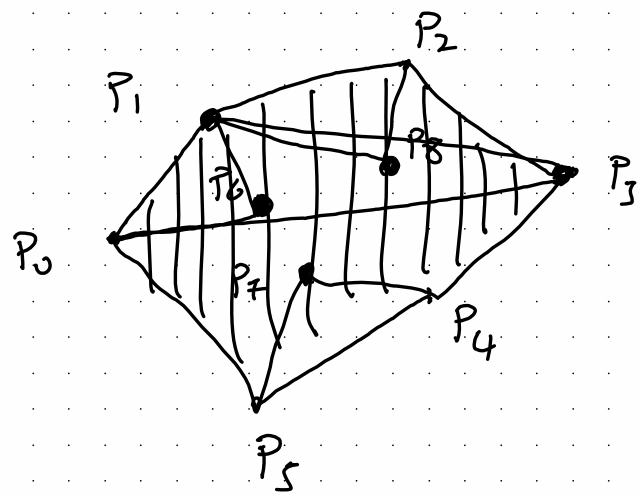
si chiama l'<u>inviluppo convesso</u> di $P_0, P_1, \dots, P_n$.

<u>Proprietà</u>:

1) $\mathrm{Conv}(P_0, \dots, P_n)$ è convesso.

2) $\mathrm{Conv}(P_0, \dots, P_n)$ è il più piccolo insieme convesso che contiene $P_0, \dots, P_n$.

<u>Es</u>:



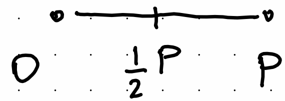
$\mathrm{Conv}(P_0, \dots, P_8)$
$= \mathrm{Conv}(P_0, \dots, P_5)$

$P_0, \dots, P_5$ sono <u>convessamente indipendenti</u>.

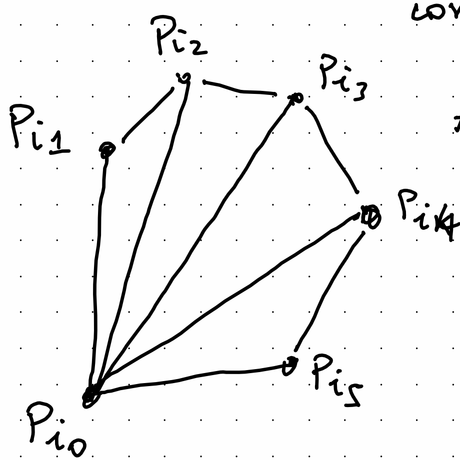
$$\mathrm{Conv}\left(0, \tfrac{1}{2}P, P\right) \qquad P \neq 0.$$

$$\| $$

$$\mathrm{Conv}(0, P)$$



$$\mathrm{Conv}(P_0, \dots, P_n) = \mathrm{Conv}(P_{i_0}, \dots, P_{i_k}) = \text{poligono}$$
con k-vertici. $= \mathcal{P}$



$$\mathrm{Area}(\mathcal{P}) = \frac{1}{2} \sum_j \left| \det\left(P_{i_j} - P_{i_0} \,\middle|\, P_{i_{j+1}} - P_{i_0}\right) \right|$$

Similmente:

$$\mathrm{Aff}(P_0, \dots, P_n) = \{ t_0 P_0 + \dots + t_n P_n \mid t_0 + t_1 + \dots + t_n = 1 \}$$

combinazioni affini di $P_0, \dots, P_n$.

$$\mathrm{Aff}(P_0, \dots, P_n) = P_0 + \langle P_1 - P_0, P_2 - P_0, \dots, P_n - P_0 \rangle$$

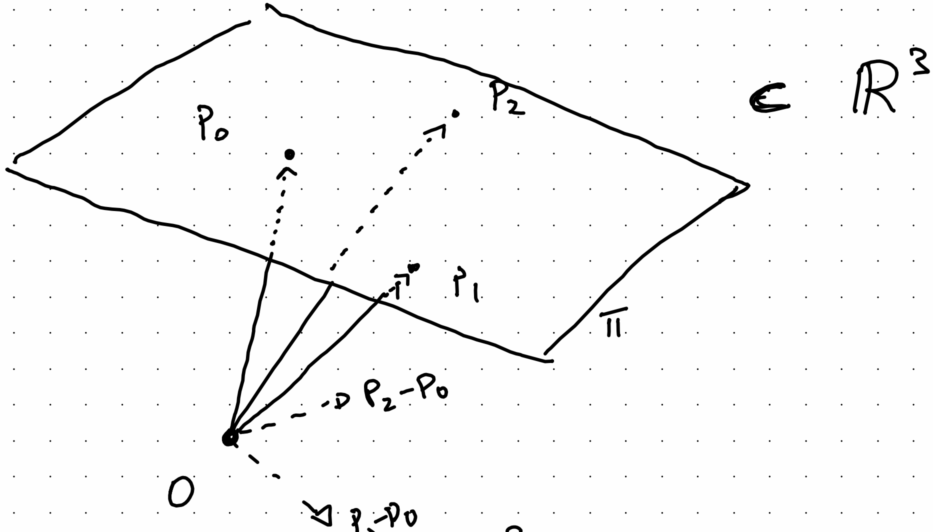
Infatti:

$$t_0 P_0 + t_1 P_1 + \dots + t_n P_n = (t_0 - 1 + 1) P_0 + t_1 P_1 + \dots + t_n P_n$$

$$= P_0 + (t_0 - 1) P_0 + t_1 P_1 + \dots + t_n P_n$$

$$= P_0 + (-t_1 - t_2 - \dots - t_n) P_0 + t_1 P_1 + \dots + t_n P_n$$

$$= P_0 + t_1 (P_1 - P_0) + t_2 (P_2 - P_0) + \dots + t_n (P_n - P_0)$$



Siano $P_0, P_1, P_2 \in \mathbb{R}^3$ distinti.

Il piano passante per P_0, P_1 e P_2 è

$$\pi = P_0 + \langle P_1 - P_0, P_2 - P_0 \rangle = \mathrm{Aff}(P_0, P_1, P_2).$$

Sistemi con parametro (Esercizi)

Studiare il seguente sistema lineare in 6 variabili $x_1, \dots, x_6$ dipendente da un parametro $k \in \mathbb{R}$:

$$\begin{cases} x_1 + (k-1)x_2 - (k-1)x_4 - (k-1)x_6 = k-3 \\ 3x_1 + 3(k-1)x_2 + x_3 - (k-1)x_4 - (k+1)x_6 = k^2+k-7 \\ x_1 + (k-1)x_2 + x_3 + (k-1)x_4 + (k-2)(k-1)x_5 + (4k-6)x_6 = k^2-3 \end{cases}$$

Sol.:

$$\left(\begin{array}{cccccc|c} 1 & k-1 & 0 & -(k-1) & 0 & -(k-1) & k-3 \\ 3 & 3(k-1) & 1 & -(k-1) & 0 & -(k+1) & k^2+k-7 \\ 1 & (k-1) & 1 & (k-1) & (k-2)(k-1) & 4k-6 & k^2-3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|c} 1 & k-1 & 0 & -k+1 & 0 & -k+1 & k-3 \\ 0 & 0 & 1 & 2(k-1) & 0 & -k-1+3k-3 & k^2+k-7-3k+9 \\ 0 & 0 & 1 & 2(k-1) & (k-2)(k-1) & 5k-7 & k^2-k \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|c} 1 & K-1 & 0 & -K+1 & 0 & -K+1 & K-3 \\ 0 & 0 & 1 & 2(K-1) & 0 & -K-1+3K-3 & K^2+K-7-3K+9 \\ 0 & 0 & 1 & 2(K-1) & (K-2)(K-1) & 5K-7 & K^2-K \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|c} 1 & K-1 & 0 & 1-K & 0 & 1-K & K-3 \\ 0 & 0 & 1 & 2K-2 & 0 & 2K-4 & K^2-2K+2 \\ 0 & 0 & 0 & 0 & (K-2)(K-1) & 3K-3 & K-2 \end{array}\right)$$

Se $K = 1$:

$$\left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{array}\right)$$

il sistema non è risolubile.

$$\left(\begin{array}{cccccc|c} 1 & K-1 & 0 & 1-K & 0 & 1-K & K-3 \\ 0 & 0 & 1 & 2K-2 & 0 & 2K-4 & K^2-2K+2 \\ 0 & 0 & 0 & 0 & (K-2)(K-1) & 3K-3 & K-2 \end{array}\right)$$

Se $K=2$:

$$\left(\begin{array}{cccccc|c} 1 & 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|c} 1 & 1 & 0 & -1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cccccc|c} 1 & 1 & 0 & -1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) \qquad \begin{cases} x_1 + x_2 - x_4 = -1 \\ x_3 + 2x_4 = 2 \\ x_6 = 0 \end{cases}$$

Le soluzioni sono

$$\begin{pmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{array}{ccc} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix}, & \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\ x_2 & x_4 & x_5 \end{array} \right\rangle$$

$$\left(\begin{array}{cccccc|c} 1 & k-1 & 0 & 1-k & 0 & 1-k & k-3 \\ 0 & 0 & 1 & 2k-2 & 0 & 2k-4 & k^2-2k+2 \\ 0 & 0 & 0 & 0 & (k-2)(k-1) & 3k-3 & k-2 \end{array}\right)$$

Se $(k-2)(k-1) \neq 0$:

$$\leadsto \left(\begin{array}{cccccc|c} 1 & k-1 & 0 & 1-k & 0 & 1-k & k-3 \\ 0 & 0 & 1 & 2k-2 & 0 & 2k-4 & k^2-2k+2 \\ 0 & 0 & 0 & 0 & 1 & \frac{3\cancel{(k-1)}}{(k-2)\cancel{(k-1)}} & \frac{\cancel{k-2}}{(k-1)\cancel{(k-2)}} \end{array}\right)$$

Le soluzioni sono

$$\begin{pmatrix} k-3 \\ 0 \\ k^2-2k+2 \\ 0 \\ 1/k-1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1-k \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k-1 \\ 0 \\ 2-2k \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} k-1 \\ 0 \\ 4-2k \\ 0 \\ -3/k-2 \\ 1 \end{pmatrix} \right\rangle$$

Es: Stabilire la posizione reciproca delle due rette di $\mathbb{R}^3$

$$r: \begin{cases} 2x+3y-z=5 \\ x+2y+z=6 \end{cases} \qquad s = \begin{pmatrix}1\\1\\1\end{pmatrix} + \left\langle \begin{pmatrix}1\\0\\1\end{pmatrix} \right\rangle$$

Sol.:

$r: \ AX=b \qquad s = X_0 + \langle v \rangle$

$$A(X_0+tv)=b \iff t\,Av = b - AX_0$$

	$rg(Av)$	$rg(Av \mid b-AX_0)$
$r \equiv s$	0	0
$r \parallel s,\ r\cap s=\phi$	0	1
$r\cap s=\{P_0\}$	1	1
sghembe.	1	2

sghembe $\Updownarrow$ non appartengono allo stesso piano.

$$\begin{pmatrix}2&3&-1\\1&2&1\end{pmatrix}\begin{pmatrix}1\\0\\1\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

$$\begin{pmatrix}5\\6\end{pmatrix}-\begin{pmatrix}2&3&-1\\1&2&1\end{pmatrix}\begin{pmatrix}1\\1\\1\end{pmatrix}=$$

$$\begin{pmatrix}5\\6\end{pmatrix}-\begin{pmatrix}4\\4\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$$

$\Rightarrow r\cap s=\{P_0\}$.

$t\begin{pmatrix}1\\2\end{pmatrix}=\begin{pmatrix}1\\2\end{pmatrix}$

$\iff t=1$

$$\Rightarrow P_0 = X_0+tv = X_0+v = \begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}.$$

Es: Stabilire la posizione reciproca di

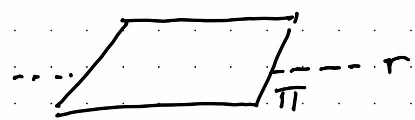
$$r: \begin{cases} x+2y-3z=2 \\ 2x+3y+z=1 \end{cases} \qquad \pi = \begin{pmatrix}1\\1\\1\end{pmatrix} + \Big\langle \underbrace{\begin{pmatrix}1\\-1\\1\end{pmatrix}}_{v_1''}, \underbrace{\begin{pmatrix}1\\2\\3\end{pmatrix}}_{v_2''} \Big\rangle$$

Sol.:

$r: AX=b$, $\pi = X_0 + \langle v_1, v_2 \rangle$ $\qquad A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 1 \end{pmatrix}$ $\qquad \operatorname{rg} A = 2$, $\dim \operatorname{Ker} A = 1$

$$A(X_0 + t_1 v_1 + t_2 v_2) = b \qquad (Av_1 | Av_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = b - AX_0$$

	$\operatorname{rg}(Av_1 \mid Av_2)$	$\operatorname{rg}(Av_1 \mid Av_2 \mid b-AX_0)$
	$\cancel{0}$	$\cancel{0}$
	$\cancel{0}$	$\cancel{1}$
$r \subset \pi$	1	1
$r \parallel \pi$, $r \cap \pi = \emptyset$	1	2
$r \cap \pi = \{P_0\}$.	2	2



$$Av_1 = \begin{pmatrix}-4\\0\end{pmatrix}, \quad Av_2 = \begin{pmatrix}-4\\11\end{pmatrix}, \quad b - AX_0 = \begin{pmatrix}2\\1\end{pmatrix} - \begin{pmatrix}0\\6\end{pmatrix} = \begin{pmatrix}2\\-5\end{pmatrix}$$

$$A v_1 = \begin{pmatrix} -4 \\ 0 \end{pmatrix}, \quad A v_2 = \begin{pmatrix} -4 \\ 11 \end{pmatrix}, \quad b - A X_0 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

$\Rightarrow \quad \pi \cap r = \{ P_0 \}. \qquad P_0 = t_1 v_1 + t_2 v_2 + X_0$ dove $\begin{pmatrix} t_1 \\ t_2 \end{pmatrix}$ è

l'unica soluzione di $(A v_1 | A v_2)\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = b - A X_0$

$$\begin{pmatrix} t_1 \\ t_2 \end{pmatrix} = (A v_1 | A v_2)^{-1} (b - A X_0)$$

$$= \begin{pmatrix} -4 & -4 \\ 0 & 11 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

$$= \frac{1}{-44} \begin{pmatrix} 11 & 4 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} 2 \\ -5 \end{pmatrix} = -\frac{1}{44} \begin{pmatrix} 2 \\ 20 \end{pmatrix} = \begin{pmatrix} -1/22 \\ -5/11 \end{pmatrix}$$

$$\Rightarrow \quad P_0 = -\frac{1}{22} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{5}{11} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 11 \\ 3 \\ -9 \end{pmatrix}$$