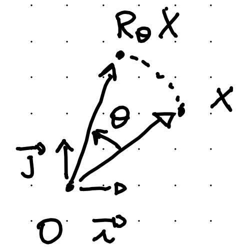
Richiami: Isometrie del piano:

1) $t_{X_0} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$X \longmapsto X + X_0$

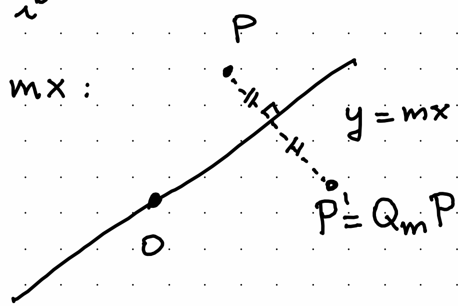
Isometrie lineari: 1) Rotazioni:

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} :$$



2) Riflessioni ortogonali rispetto a $y = mx$:

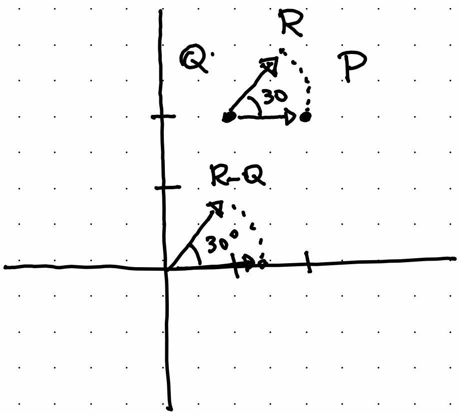
$$Q_m = \frac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$



$$Q_\infty = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad : \text{ rispetto a } x = 0$$

Es: Sia $Q = \begin{pmatrix}1\\2\end{pmatrix}$. Trovare il punto R ottenuto ruotando di $30°$ in senso anti-orario il punto $P = \begin{pmatrix}2\\2\end{pmatrix}$ attorno al punto Q.

Sol.:



$$R - Q = R_{30°}(P - Q)$$

$$\Rightarrow R = Q + R_{30°}(P-Q)$$

$$= \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \begin{pmatrix}1\\2\end{pmatrix} + \begin{pmatrix}\sqrt{3}/2\\ 1/2\end{pmatrix} = \begin{pmatrix}1 + \frac{\sqrt{3}}{2}\\ \frac{5}{2}\end{pmatrix}$$

Es: Sia $Q = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$. Trovare il punto ottenuto riflettendo ortogonalmente Q attraverso la retta

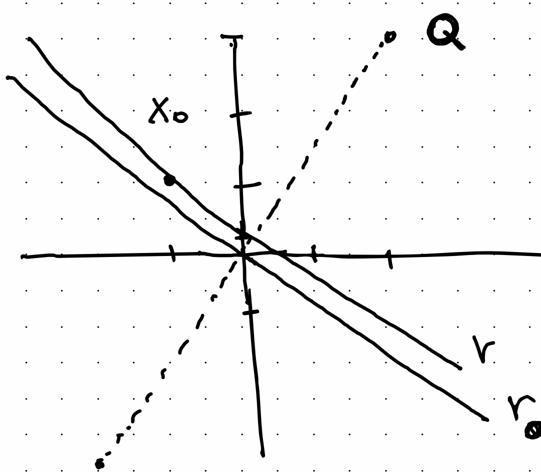
$$r: \quad 2x+3y=1$$

Sol.:

$$r = X_0 + r_0 \quad \text{dove} \quad X_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad r_0: 2x+3y=0$$

Il punto cercato R è t.c.

$$R - X_0 = Q_m (Q - X_0), \quad r_0: \ y = -\frac{2}{3}x$$



$$Q - X_0 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$$

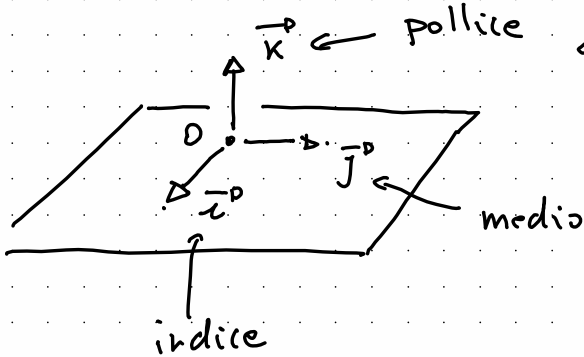
$$Q_m = Q_{-\frac{2}{3}} = \frac{1}{1+\frac{4}{9}} \begin{pmatrix} \frac{5}{9} & -\frac{4}{3} \\ -\frac{4}{3} & -\frac{5}{9} \end{pmatrix}$$

$$= \frac{1}{13} \begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}$$

$$R = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \frac{1}{13} \begin{pmatrix} -22 \\ -33 \end{pmatrix}$$

Geometria analitica dello spazio

Base standard di V_0^3: $\mathcal{B} = \{\vec{i}, \vec{j}, \vec{k}\}$



regola della mano destra

$|\vec{i}| = |\vec{j}| = |\vec{k}| = 1$

$\vec{i} \perp \vec{j} \perp \vec{k} \perp \vec{i}$.

$$F_{\mathcal{B}} : V_0^3 \xrightarrow{\simeq} \mathbb{R}^3$$

prodotto scalare standard di V_0^3.

$$F_{\mathcal{B}}(v) \cdot F_{\mathcal{B}}(w) = v \cdot w \overset{\checkmark}{=} |v||w| \cos\vartheta$$

Una base $\mathcal{B} = \{v_1, v_2, v_3\}$ di $\mathbb{R}^3$ è equiversa se

$$\det(v_1|v_2|v_3) > 0$$

altrimenti si dice contraversa.

Il <u>prodotto vettoriale</u> o <u>prodotto vettore</u>

$$\wedge : \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(v, w) \longmapsto v \wedge w$$

dato da

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \wedge \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} \det \begin{pmatrix} x_2 & y_2 \\ x_3 & y_3 \end{pmatrix} \\ -\det \begin{pmatrix} x_1 & y_1 \\ x_3 & y_3 \end{pmatrix} \\ \det \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \end{pmatrix} \end{pmatrix}$$

<u>Es</u>:

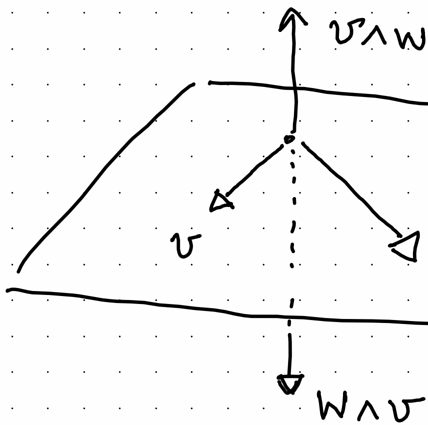
$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ -5 \end{pmatrix} \quad , \quad v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad w = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \quad u = \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

$$(v \wedge w) \cdot u = \det \begin{pmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{pmatrix}$$

<u>Prodotto misto</u> di u, v, w

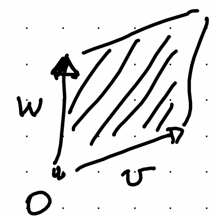
$$u \cdot (v \wedge w) \in \mathbb{R}.$$

Proprietà: 1) $\{v, w, v\wedge w\}$ è equiversa:



2) $v\wedge w \cdot v = 0 = v\wedge w \cdot w$

$(v\wedge w)$ è ortogonale a $\{v, w\}$.

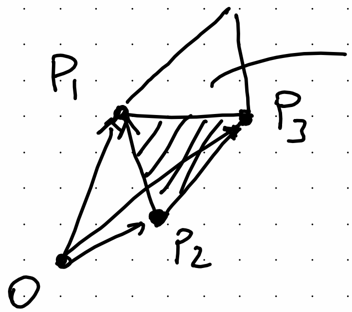
3) $\|v\wedge w\| =$ Area Parallelogramma  $=$

$=$ Area $P(v,w)$.

In particolare, $v\wedge w = 0_{\mathbb{R}^3} \iff v$ e w sono lin. Dip.

Es: Calcolare l'area del triangolo di vertici

$$P_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \quad P_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \quad P_3 = \begin{pmatrix} 2 \\ 3 \\ 2 \end{pmatrix}$$



$$\text{Area } \mathcal{P}(P_3 - P_2, P_1 - P_2) = $$
$$= \| (P_3 - P_2) \wedge (P_1 - P_2) \|$$

$$\text{Area } \widehat{P_1 P_2 P_3} = \frac{1}{2} \| (P_3 - P_2) \wedge (P_1 - P_2) \|$$

$$= \frac{1}{2} \left\| \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix} \right\|$$

$$= \frac{1}{2} \left\| \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix} \right\| = \frac{1}{2} \sqrt{1+4+1} = \frac{\sqrt{6}}{2}$$

OSS (importante!):

Sia

$$r : \begin{cases} ax+by+cz=d \\ a'x+b'y+c'z=d' \end{cases} \qquad r_0 : \begin{cases} ax+by+cz=0 \\ a'x+b'y+c'z=0 \end{cases}$$

$$r_0 = \left\{ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{cases} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot X = 0 \\ \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \cdot X = 0 \end{cases} \right\}$$

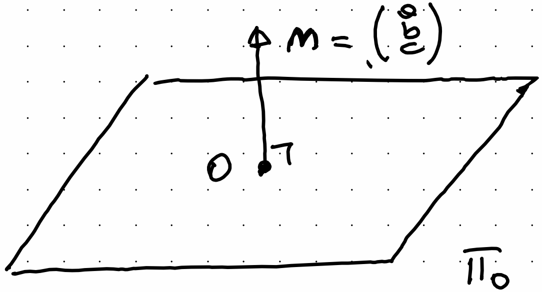
$$r_0^{\perp} = \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \right\rangle .$$

Sia $\pi : ax+by+cz=d$ un piano di $\mathbb{R}^3$.

$\pi_0 : ax+by+cz=0$

$$\pi_0^\perp = \left\langle \begin{pmatrix} a \\ b \\ c \end{pmatrix} \right\rangle$$

$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$ = vettore normale al piano



$\pi : \quad n \cdot X = d$

$\pi_0 = \langle v, w \rangle \implies n = v \wedge w \implies \pi_0 : (v \wedge w) \cdot X = 0$

Es: Trovare eq. cartesiana del piano

$$\pi = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} \right\rangle$$

Sol.:

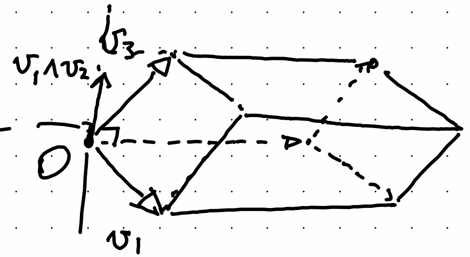
$$n = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} -1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix}$$

$$\pi : \quad n \cdot X = n \cdot X_0 = \begin{pmatrix} 1 \\ -3 \\ 5 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 1 - 3 + 5 = 3$$

$$\Rightarrow \quad \pi : \quad x - 3y + 5z = 3.$$

Il determinante 3x3 come volume (orientato)

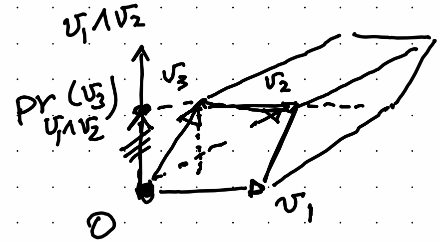
$v_1, v_2, v_3 \in \mathbb{R}^3$. Sia $P(v_1, v_2, v_3)$ il parallelepipedo che ha come spigoli v_1, v_2 e v_3:



$= P(v_1, v_2, v_3)$.

<u>Teorema:</u>

$$\mathrm{Vol}\, P(v_1, v_2, v_3) = |\det(v_1 | v_2 | v_3)|$$



<u>dim:</u>

$$\mathrm{Vol}\, P(v_1, v_2, v_3) = \text{Area di base} \times \text{Altezza}$$

$$= \|v_1 \wedge v_2\| \, \|pr_{v_1 \wedge v_2}(v_3)\|$$

$$= \|v_1 \wedge v_2\| \frac{|(v_1 \wedge v_2) \cdot v_3|}{\|v_1 \wedge v_2\|} = |(v_1 \wedge v_2) \cdot v_3| = |\det(v_1 | v_2 | v_3)|$$

$\blacksquare$

Vettore direttore di r:

$$r : \begin{cases} a\,x + by + cz = d \\ a'x + b'y + c'z = d' \end{cases}$$

Qual'è un vettore direttore di r? ovvero una soluzione del sistema omogeneo associato

$$(*) \quad \begin{cases} ax + by + cz = 0 \\ a'x + b'y + c'z = 0 \end{cases} \quad ?$$

$$(*) \iff \begin{cases} \begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot X = 0 \\ \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \cdot X = 0 \end{cases} \iff X : X \text{ è ortogonale a } \begin{pmatrix} a \\ b \\ c \end{pmatrix} \text{ e } X \text{ è ortogonale a } \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}.$$

$$X = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \wedge \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix}$$

Es:

$$r: \begin{cases} 2x+3y-z=3 \\ x-y+2z=1 \end{cases}$$

Un vettore direttore per r è

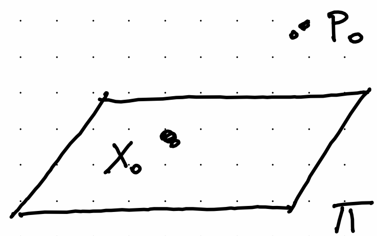
$$v = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ -5 \\ -5 \end{pmatrix}$$

Distanza punto-piano

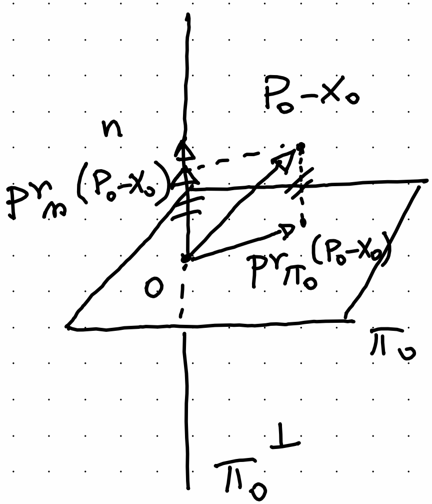
Siano $P_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in \mathbb{R}^3$ e $\pi \subset \mathbb{R}^3$ un piano.

$\pi = X_0 + \pi_0$. Sia n un vettore normale a π_0

$\pi_0 : \; n \cdot X = 0$, $\quad \pi : \; n \cdot X = n \cdot X_0$.



$\pi : \; ax + by + cz = d$



$$\mathrm{dist}(P_0, \pi) = \mathrm{dist}(P_0 - X_0, \pi_0)$$

$$= \| pr_n (P_0 - X_0) \|$$

$$= \frac{| n \cdot (P_0 - X_0) |}{\| n \|} = \frac{| a x_0 + b y_0 + c z_0 - d |}{\sqrt{a^2 + b^2 + c^2}}$$

Se $\pi = X_0 + \langle v_1, v_2 \rangle$

$$\mathrm{dist}(P_0, \pi) = \| pr_{v_1 \wedge v_2}(P_0 - X_0) \|$$

$$= \frac{|(v_1 \wedge v_2) \cdot (P_0 - X_0)|}{\| v_1 \wedge v_2 \|}$$

$$= \frac{|\det(v_1 | v_2 | P_0 - X_0)|}{\| v_1 \wedge v_2 \|}$$

Es: $\pi : 2x + y - z = 2 \qquad P_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$$\mathrm{dist}(P_0, \pi) = \frac{|2 + 1 - 1 - 2|}{\sqrt{2^2 + 1^2 + (-1)^2}} = 0 \quad (\text{infatti, } P_0 \in \pi).$$

$$\pi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\rangle \Rightarrow \mathrm{dist}(P_0, \pi) = \frac{\left| \det \begin{pmatrix} 1 & 1 & 0 \\ -2 & 0 & 1 \\ 0 & 2 & 1 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} \right\|} =$$

$$= \frac{0}{\left\| \begin{pmatrix} -4 \\ -2 \\ 2 \end{pmatrix} \right\|} = \frac{0}{2 \left\| \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix} \right\|} = \frac{0}{2\sqrt{6}} = 0$$

Distanza retta-retta

$r = X_0 + \langle v \rangle \quad , \quad s = Y_0 + \langle w \rangle$

$$\mathrm{dist}(r,s) = \min_{h,k \in \mathbb{R}} \mathrm{dist}(X_0 + hv, Y_0 + kw)$$

$$= \min_{h,k \in \mathbb{R}} \mathrm{dist}(X_0 - Y_0, kw - hv)$$

$$= \mathrm{dist}(X_0 - Y_0, \langle v, w \rangle)$$

$$= \frac{|(X_0 - Y_0) \cdot v \wedge w|}{\| v \wedge w \|}$$

$$= \frac{|\det(v | w | X_0 - Y_0)|}{\| v \wedge w \|}$$

Es:

$$r_1: \begin{cases} x+y-z=2 \\ 2x-y+2z=1 \end{cases} \qquad r_2: \begin{cases} x-2y-2z=1 \\ 2x+y-z=2 \end{cases}$$

Calcolare $\mathrm{dist}(r_1, r_2)$.

Sol.:

$$r_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix} \right\rangle = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \right\rangle$$

$$r_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} \right\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} \right\rangle$$

$$\mathrm{dist}(r_1, r_2) = \frac{\left| \det \begin{pmatrix} 1 & 4 & 0 \\ -4 & -3 & 1 \\ -3 & 5 & 0 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 1 \\ -4 \\ -3 \end{pmatrix} \wedge \begin{pmatrix} 4 \\ -3 \\ 5 \end{pmatrix} \right\|} = \frac{|-17|}{\left\| \begin{pmatrix} -29 \\ -17 \\ 13 \end{pmatrix} \right\|} = \frac{17}{\sqrt{1299}}$$

Distanza punto - retta

Sia $P_0 = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} \in \mathbb{R}^3$, $\quad r = X_0 + r_0 = X_0 + \langle v_0 \rangle$

$$\operatorname{dist}(P_0, r) = \operatorname{dist}(P_0 - X_0, r_0)$$

$$= \| (P_0 - X_0) - \operatorname{pr}_{r_0}(P_0 - X_0) \|$$

$$= \left\| P_0 - X_0 - \frac{v_0 \cdot (P_0 - X_0)}{v_0 \cdot v_0} v_0 \right\|$$

<u>Es</u>: $P_0 = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \qquad r = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\rangle$

$$\operatorname{dist}(P_0, r) = \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - \frac{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}}{\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} - \frac{-1}{9} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} \right\| = \left\| \begin{pmatrix} 2/9 \\ -8/9 \\ 2/9 \end{pmatrix} \right\| = \frac{2}{9} \left\| \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix} \right\|$$

$$= \frac{2}{9} \sqrt{1 + 16 + 1} = \frac{2\sqrt{18}}{9} = \frac{2}{3}\sqrt{2}$$

Diagonalizzazione di endomorfismi lineari

Def: Sia V uno spazio vettoriale su un campo $\mathbb{K}$.
Sia $\mathcal{L}: V \to V$ un endomorfismo lineare.
Si dice che $\mathcal{L}$ è diagonalizzabile se esiste una base $\mathcal{B} = \{v_1, \dots, v_n\}$ di V nella quale, la matrice che rappresenta $\mathcal{L}$ (sia in partenza che in arrivo) è diagonale

$$D = \mathrm{diag}(\lambda_1, \dots, \lambda_n) = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

$$\begin{array}{ccc} V & \xrightarrow{\mathcal{L}} & V \\ {\scriptstyle F_{\mathcal{B}}}\downarrow & & \downarrow{\scriptstyle F_{\mathcal{B}}} \\ \mathbb{K}^n & \xrightarrow[D]{} & \mathbb{K}^n \end{array}$$

ovvero $\boxed{\mathcal{L}(v_i) = \lambda_i v_i}$ $\quad \forall i = 1, \dots, n.$

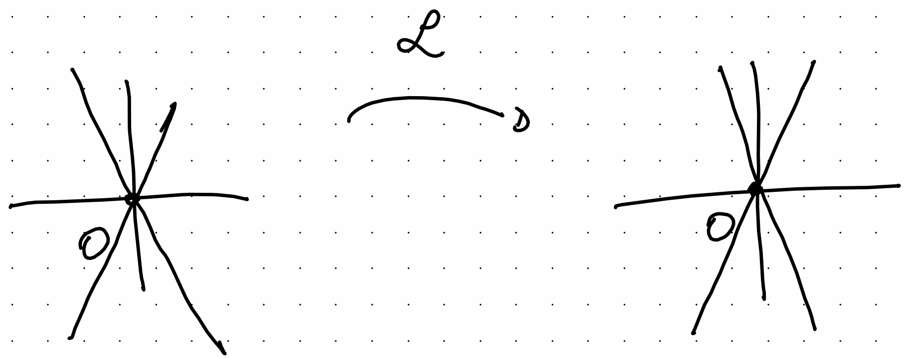
$\mathcal{B}$ si chiama base diagonalizzante.

Se $\mathcal{B} = \{v_1, \dots, v_n\}$ è una base diagonalizzante per $\mathcal{L}$ allora

$$\mathcal{L}\langle v_i\rangle \subseteq \langle v_i\rangle \qquad (*)$$

$$\left(\mathcal{L}(tv_i) = t\,\mathcal{L}(v_i) = t\,\lambda_i v_i \in \langle v_i \rangle \right)$$

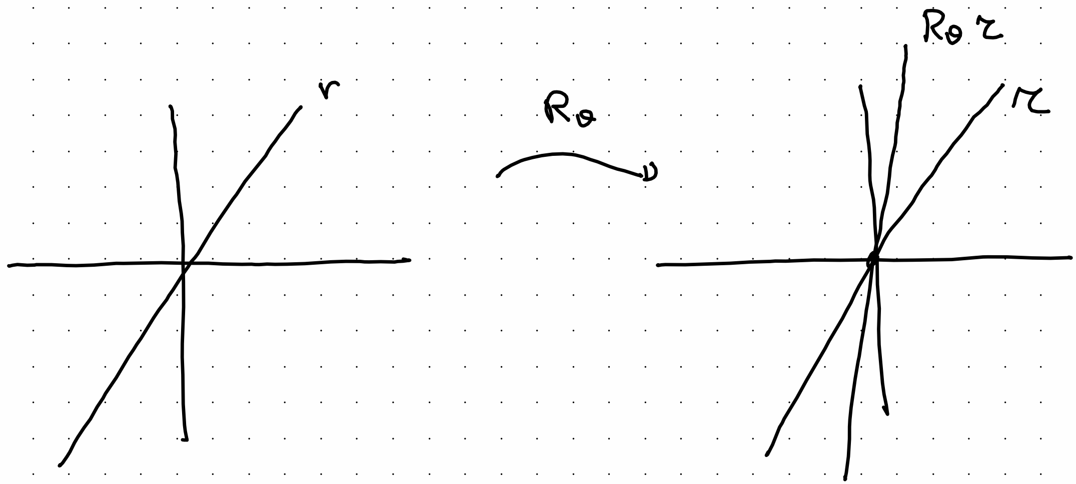
Le n rette $\langle v_1\rangle, \dots, \langle v_n\rangle$ sono *invarianti* ovvero $(*)$.



Assi di simmetria per $\mathcal{L}$.

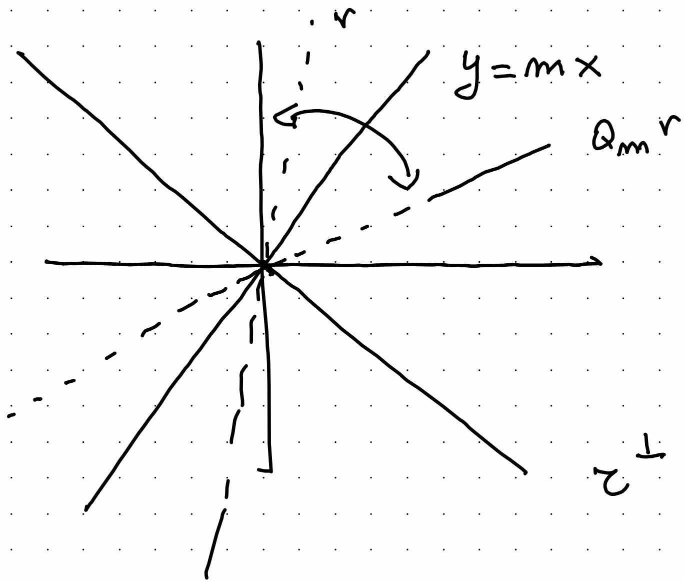
Es: $\mathcal{L} = R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ è diagonalizzabile?

Ci sono due rette fissate da R_θ?



$\exists$ due rette distinte fissate $\Leftrightarrow$ $\theta = 0$ oppure $\theta = \pi$.

Es: $\mathcal{L} = Q_m = \dfrac{1}{1+m^2}\begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$



$\left.\begin{array}{l} r:\ y = mx \ \text{è fissata da } Q_m. \\ r^\perp \ \text{è fissata da } Q_m \end{array}\right\} \Rightarrow$ Q_m è diagonalizzabile.