Es 5: $p(x,y) = 5x^2+4xy+2y^2-2x+4y-1$

Trovare la forma canonica metrica e affine di $\mathcal{C}_p = \{X \in \mathbb{R}^2 \mid p(X)=0\}$.

Sol.: $A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$ $\quad b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$ $\quad f = -1$

$$\hat{A} = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & -1 \end{pmatrix} = \left(\begin{array}{c|c} A & b \\ \hline b^t & f \end{array}\right)$$

$\operatorname{Tr} A = 7 > 0$

$\det A = 6 > 0$

$\det \hat{A} = -36 < 0$

OSS: $q(X) = X^t A X$ forma quadratica in 2 variabili. $Sp(A) = \{\lambda_1 \geq \lambda_2\}$

Forma canonica di Sylvester	$sg(A)$	$\operatorname{Tr}(A) = \lambda_1+\lambda_2$	$\det A = \lambda_1\lambda_2$
x^2+y^2	$(2,0)$	>0	>0
x^2-y^2	$(1,1)$	?	<0
x^2	$(1,0)$	>0	0
$-y^2$	$(0,1)$	<0	0
$-x^2-y^2$	$(0,2)$	<0	>0

$$\left.\begin{array}{l} sg(A) = (2,0) \\ sg(\hat{A}) = (2,1) \end{array}\right] \Rightarrow p \text{ è un ellisse reale.}$$

$x^2+y^2-1=0$ Forma can. affine.

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\hat{A} = \begin{pmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & -1 \end{pmatrix} \qquad f = -1.$$

$$p(x) = x^t A x + 2\, b \cdot x + f$$

1) Cerchiamo B ortogonale t.c. $B^t A B$ è diagonale.

$$p_A(x) = x^2 - 7x + 6 = (x-1)(x-6)$$

Autospazi:

$$V_1(A) = \operatorname{Ker}(\mathbb{1}_2 - A) = \operatorname{Ker}\begin{pmatrix} -4 & -2 \\ -2 & -1 \end{pmatrix} = \operatorname{Ker}(2\ 1) = \left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle$$

$$V_6(A) = \operatorname{Ker}(6\mathbb{1}_2 - A) = \operatorname{Ker}\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} = \operatorname{Ker}(1, -2) = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle$$

$$B = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix}$$

Poniamo $\boxed{X = BY}$ ovvero $\begin{cases} x_1 = \frac{-1}{\sqrt{5}} y_1 + \frac{2}{\sqrt{5}} y_2 \\ x_2 = \frac{2}{\sqrt{5}} y_1 + \frac{1}{\sqrt{5}} y_2 \end{cases}$

e otteniamo

$$p(x) = Y^t D Y + 2\, b \cdot BY + f$$

$$= Y^t D Y + 2\, B^t b \cdot Y + f =$$

$$B^t b = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 5 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix}$$

$$= y_1^2 + 6 y_2^2 + \underline{2\sqrt{5}\, y_1} - 1$$

2) Cerchiamo, se esiste, un vettore C t.c. $q(Z+C)$ non ha termine lineare:

$$q(Z+C) = (Z+C)^t D (Z+C) + 2 B^t b \cdot (Z+C) + f$$

$$= \underline{Z^t D Z} + Z^t DC + \underline{C^t D Z} + \underline{C^t DC} +$$

$$+ 2 \underline{B^t b \cdot Z} + 2 B^t b \cdot C + f$$

$$= Z^t D Z + 2 (DC + B^t b) \cdot Z + q(C)$$

Cerchiamo C t.c. $DC + B^t b = 0$

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 6 \end{pmatrix} \qquad B^t b = \begin{pmatrix} \sqrt{5} \\ 0 \end{pmatrix}$$

Una soluzione C di

$$DC = -B^t b$$

è

$$C = \begin{pmatrix} -\sqrt{5} \\ 0 \end{pmatrix}$$

$$Y = Z + C \quad \text{ovvero} \quad \begin{cases} y_1 = z_1 - \sqrt{5} \\ y_2 = z_2 \end{cases}$$

otteniamo

$$p(X) = q(Y) = r(Z) = z_1^2 + 6 z_2^2 + q(C) \ldots$$

Es 1:

1) $R = P + R_{\frac{\pi}{6}}(Q-P) = \begin{pmatrix}2\\2\end{pmatrix} + \begin{pmatrix}\sqrt{3}/2 & -1/2\\ 1/2 & \sqrt{3}/2\end{pmatrix}\begin{pmatrix}-1\\0\end{pmatrix} = \frac{1}{2}\begin{pmatrix}4-\sqrt{3}\\3\end{pmatrix}$

2) $r_0 : \; y = -\frac{2}{3}x \;, \; R = X_0 + Q_{-\frac{2}{3}}(P - X_0) = \frac{1}{13}\begin{pmatrix}-22\\-33\end{pmatrix}$

3) $2(A-B)\cdot X = A^2 - B^2$

$2\begin{pmatrix}-4\\3\end{pmatrix}\cdot X = 5 - 10$

$\Rightarrow \quad -8x + 6y = -5 \iff 8x - 6y = 5$: Asse,

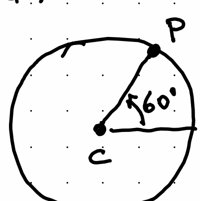
$\begin{pmatrix}0\\-5/6\end{pmatrix} + \left\langle \begin{pmatrix}6\\8\end{pmatrix} \right\rangle$

4) Completando i quadrati otteniamo

$\mathcal{C} : \left(x - \frac{3}{2}\right)^2 + (y+2)^2 = \frac{9}{4} \qquad C = \begin{pmatrix}3/2\\-2\end{pmatrix} \quad r = \frac{3}{2}$

$\mathcal{C} = \left\{ \begin{pmatrix}3/2\\-2\end{pmatrix} + \frac{3}{2}\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix} \;\middle|\; \theta \in [0, 2\pi) \right\}$

$P - C = \begin{pmatrix}3/4\\ 3\sqrt{3}/4\end{pmatrix} = \frac{3}{2}\begin{pmatrix}1/2\\ \sqrt{3}/2\end{pmatrix} \Rightarrow P = C + r\begin{pmatrix}\cos \pi/3\\ \sin \pi/3\end{pmatrix}$



$t_P = P + \left\langle \begin{pmatrix}-\sin \pi/3\\ \cos \pi/3\end{pmatrix} \right\rangle \; : \; x + \sqrt{3}y = \frac{9}{2} - 2\sqrt{3}$

5) $\text{dist}(P, z) = \frac{|2\cdot 3 - 2 - 1|}{\sqrt{5}} = \frac{3}{\sqrt{5}}$

6) $1 = \dfrac{|-2+m|}{|1+2m|} \iff -2+m = \pm(1+2m)$

$\iff m = 3$ oppure $m = \dfrac{1}{3}$

le cercate sono:

$r_1 : \quad -3x + y = -4 \qquad r_1 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\rangle$

$r_2 : \quad x + 3y = 8 \qquad r_2 = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1/3 \end{pmatrix} \right\rangle$

$P_1 = r \cap r_1 = \begin{pmatrix} 2 & 1 \\ -3 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$P_2 = r \cap r_2 = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 8 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$\text{Area}\,(\widehat{P P_1 P_2}) = \dfrac{1}{2} \left| \det (P - P_1, P_2 - P_1) \right| = 5$

$\text{Perimetro}\,(\widehat{P P_1 P_2}) = \|P - P_1\| + \|P - P_2\| + \|P_1 - P_2\| = 2\sqrt{5}\,(1 + \sqrt{2})$

Es 2:

1. $\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow$ r ed s non parallele

$\begin{pmatrix} 2 & 3 & -1 \\ 1 & 2 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix}. \quad b - \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \end{pmatrix} - \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\Rightarrow r \cap s = \{P_0\} \quad P_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$

2. $\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} -4 & -4 \\ 0 & 11 \end{pmatrix}$

$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 6 \end{pmatrix}$

$b - \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 6 \end{pmatrix} = \begin{pmatrix} 2 \\ -5 \end{pmatrix}$

$\left(\begin{array}{cc|c} -4 & -4 & 2 \\ 0 & 11 & -5 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & -1/22 \\ 0 & 1 & -5/11 \end{array}\right) \Rightarrow P_0 = r \wedge \pi =$

$= -\frac{1}{22} \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} - \frac{5}{11} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 11 \\ 3 \\ -9 \end{pmatrix}$

3) $\pi = P_1 + \langle P_2 - P_1, P_3 - P_1 \rangle = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle : z = 1$

4) $\det \begin{pmatrix} 1 & 1 & 3 \\ 1 & -2 & -6 \\ 0 & 1 & 0 \end{pmatrix} = -\det \begin{pmatrix} 1 & 3 \\ 1 & -6 \end{pmatrix} = 9 \neq 0 \Rightarrow$ r ed s sono sghembe. $r: \begin{cases} x - y = 4 \\ z = 2 \end{cases}$

$\pi_{\alpha,\beta}: \alpha(x - y - 4) + \beta(z - 2) = 0$. s è parallela a $\pi_{\alpha,\beta}$

$\Leftrightarrow \quad (\alpha, -\alpha, \beta) \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = 3\alpha + \beta = 0$

$\Leftrightarrow \quad \beta = -3\alpha$

$\pi = \pi_{1,-3}: \quad x - y - 4 + 3(z - 2) = 0 \Leftrightarrow \pi: x - y - 3z = -2$

5) $P_2 - P_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} \qquad P_3 - P_1 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix}$

$$\text{Area} = \frac{1}{2} \| (P_2 - P_1) \wedge (P_3 - P_1) \| = \frac{1}{2} \left\| \begin{pmatrix} 0 \\ -3 \\ -6 \end{pmatrix} \right\| = \frac{3}{2}\sqrt{5}$$

6) $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix}$

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 0 \\ -1 \\ -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ -5 \end{pmatrix} = 3 + 10 = 13$$

$$\text{dist}(r, s) = \frac{13}{\left\| \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} \right\|} = \frac{13}{\sqrt{35}}$$

7) $Q = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\text{dist}(P, \pi) = \left\| pr_{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}} (P - Q) \right\| = \frac{\left| \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right|}{\left\| \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} \right\|} = \frac{3}{\sqrt{14}}$$

Es:

$$A_K = \begin{pmatrix} -K & -K & 1 \\ K-1 & K-1 & -1 \\ K+1 & K & 0 \end{pmatrix}$$

$$P_{A_K}(x) = \det(x\mathbb{1}_3 - A_K)$$

$$= \det \begin{pmatrix} x+K & K & -1 \\ -K+1 & x-K+1 & 1 \\ -K-1 & -K & x \end{pmatrix}$$

$$= \det \begin{pmatrix} x & K & -1 \\ -x & x-K+1 & 1 \\ -1 & -K & x \end{pmatrix}$$

$$= \det \begin{pmatrix} x & K & -1 \\ 0 & x+1 & 0 \\ -1 & -K & x \end{pmatrix}$$

$$= (x+1)\det \begin{pmatrix} x & -1 \\ -1 & x \end{pmatrix}$$

$$= (x+1)(x^2-1)$$

$$= (x+1)(x+1)(x-1) =$$

$$= (x+1)^2(x-1) \Rightarrow Sp(A) = \{1, -1\}$$

$$ma_A(-1) = 2 \quad , \quad ma_A(1) = 1$$

$$A_k = \begin{pmatrix} -k & -k & 1 \\ k-1 & k-1 & -1 \\ k+1 & k & 0 \end{pmatrix}$$

$$V_{-1}(A_k) = \operatorname{Ker}\left(-\mathbb{1}_3 - A_k\right)$$

$$= \operatorname{Ker}\begin{pmatrix} -1+k & k & -1 \\ 1-k & -k & 1 \\ -1-k & -k & -1 \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ -1-k & -k & -1 \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 0 & 1 \\ -1-k & -k & -1 \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 0 & 1 \\ 0 & -k & -1+1+k \end{pmatrix}$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & 0 & 1 \\ 0 & -k & k \end{pmatrix}$$

$$= \begin{cases} \left\langle \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\rangle & \text{se } k \neq 0 \\ \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle & \text{se } k = 0 \end{cases}$$

$mg_{A_k}(-1) = 2 \iff k = 0.$

Quindi A_k è diagonalizzabile se e solo se $k = 0$.

$$A_0 = \begin{pmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$$

2) $A_k = A_k^t \iff \begin{cases} -k = k-1 \rightsquigarrow k = \frac{1}{2} \\ k = -1 \rightsquigarrow k = -1 \\ k+1 = 1 \rightsquigarrow k = 0 \end{cases}$ ↯.

A_k non è ortogonalmente diagonalizzabile per ogni k.

3) $V_{-1}(A_0) = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$

$$V_1(A_0) = \ker(\mathbb{1}_3 - A_0) = \ker \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ -1 & 0 & 1 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \end{pmatrix} = \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 2 & 2 \end{pmatrix}$$

$$= \ker \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \right\rangle$$

$$B = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix} \quad D = \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}.$$

Es4:

1) $$\det(v_1|v_2|v_3) = \det\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$= \det\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = -\det\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = -1 \neq 0$$

$\Rightarrow \mathcal{B}$ è una base

2) $$A = \begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 5 \\ -1 & 4 & -6 \end{pmatrix}$$

3) $$\begin{tikzcd}
\mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_e"'] & \mathbb{R}^3 \arrow[r, "f"] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[d, "F_e"] \\
\mathbb{R}^3 & \mathbb{R}^3 \arrow[l, "B"'] \arrow[r, "A"] & \mathbb{R}^3 \arrow[r, "B"] & \mathbb{R}^3
\end{tikzcd}$$

$B = (v_1|v_2|v_3)$. $\qquad C = BAB^{-1}$

$$(BA \mid B) \rightsquigarrow (\mathbb{1}_3 \mid C).$$

$$BA = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -1 \\ 2 & -1 & 5 \\ -1 & 4 & -6 \end{pmatrix} = \begin{pmatrix} 2 & 6 & -2 \\ 0 & 7 & -7 \\ 1 & 3 & -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 2 & 6 & -2 & 1 & 1 & 1 \\ 0 & 7 & -7 & 1 & 0 & 1 \\ 1 & 3 & -1 & 0 & 1 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 10 & -8 & -4 \\ 0 & 1 & 0 & 14 & -14 & -7 \\ 0 & 0 & 1 & 5 & -4 & -2 \end{array}\right)$$

C

4) $\operatorname{Ker} f = \operatorname{Ker} C = \operatorname{Ker} \begin{pmatrix} 10 & -8 & -4 \\ 14 & -14 & -7 \\ 5 & -4 & -2 \end{pmatrix}$

$= \dots = \operatorname{Ker} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1/2 \\ 0 & 0 & 0 \end{pmatrix} = \left\langle \begin{pmatrix} 0 \\ -1 \\ 2 \end{pmatrix} \right\rangle$

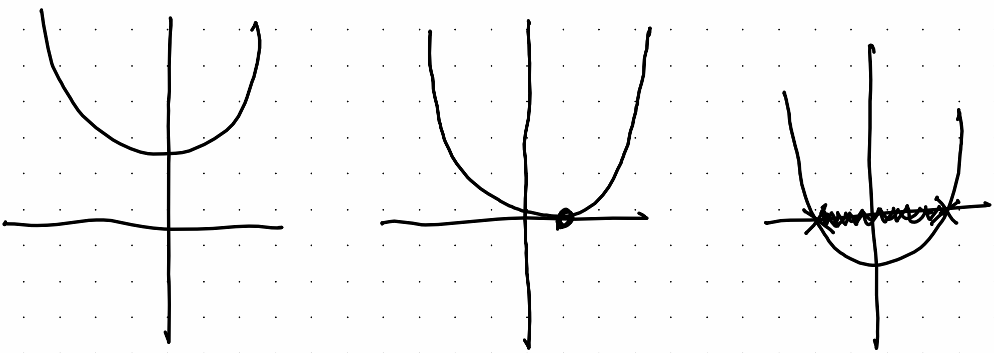
5) $\operatorname{Im} f = \operatorname{Im} C = \langle C^1, C^2 \rangle$

$= \left\langle \begin{pmatrix} 10 \\ 14 \\ 5 \end{pmatrix}, \begin{pmatrix} -8 \\ -14 \\ -4 \end{pmatrix} \right\rangle.$

Dis. Cauchy-Schwarz:

$$\| \ \| = a\lambda^2 + b\lambda + c \geq 0$$

$$\lambda_{1,2} = \frac{-b \pm \sqrt{\Delta}}{2} \quad , \quad a \geq 0 \qquad a = \|v\|$$



$\Delta < 0$

No radici reali

$\Delta = 0$

$\Delta > 0$

due radici reali