Funzioni invertibili

Una f.ne $f : X \to Y$ si dice invertibile se
$\exists\ g, h : Y \to X$ t.c.

$$g \circ f = Id_X \quad e \quad f \circ h = Id_Y .$$

Prop.: f invertibile $\Leftrightarrow$ f è biiettiva, i.e. iniettiva + suriettiva.

Oss: f invertibile $\Rightarrow$ $g = h$ ed è unica e si denota con f^{-1}.

Se $f : V_1 \to V_2$ è lineare e biiettiva allora f si dice un isomorfismo lineare.

f isomorfismo lineare $\Leftrightarrow$ $Ker f = \{0_{V_1}\}$ e $Im f = V_2$
$\Leftrightarrow$ $\dim V_1 = \dim V_2$ e $Ker f = \{0_{V_1}\}$.
$\Leftrightarrow$ f manda basi in basi,

Es: $f: \mathbb{R}^3 \to \mathbb{R}^3$ lineare

$$v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix} \longmapsto \begin{pmatrix}1\\0\\1\end{pmatrix} = w_1 = e_1 + e_3$$

$$v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix} \longmapsto \begin{pmatrix}1\\1\\0\end{pmatrix} = w_2$$

$$v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix} \longmapsto \begin{pmatrix}0\\1\\1\end{pmatrix} = w_3$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[d, "F_{\mathcal{B}}"'] & \mathbb{R}^3 \arrow[d, "F_{f(\mathcal{B})}"] \\ \mathbb{R}^3 \arrow[r] & \mathbb{R}^3 \end{tikzcd}$$

$$A = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} = \mathbb{1}_3$$

$\mathcal{B} = \{v_1, v_2, v_3\}$ è una base di $\mathbb{R}^3$.

$f(\mathcal{B}) = \{w_1, w_2, w_3\}$ è una base di $\mathbb{R}^3 \Rightarrow f$ è un isomorfismo.

Chi è f^{-1}:

$$w_1 = \begin{pmatrix}1\\0\\1\end{pmatrix} \longmapsto v_1 = \begin{pmatrix}1\\1\\0\end{pmatrix}$$

$$w_2 = \begin{pmatrix}1\\1\\0\end{pmatrix} \longmapsto v_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$$

$$w_3 = \begin{pmatrix}0\\1\\1\end{pmatrix} \longmapsto v_3 = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "f"] \arrow[d, "F_{\mathcal{B}}"'] & \mathbb{R}^3 \arrow[d, "F_{e}"] \\ \mathbb{R}^3 \arrow[r] & \mathbb{R}^3 \end{tikzcd}$$

$$\begin{pmatrix}1&1&0\\0&1&1\\1&0&1\end{pmatrix}$$

$$
\begin{tikzcd}
V \arrow[r, "Id_V"] \arrow[d, "F_{\mathcal{B}_1}"'] & V \arrow[d, "F_{\mathcal{B}_2}"] \\
\mathbb{K}^n \arrow[r, "B"'] & \mathbb{K}^n
\end{tikzcd}
$$

$\mathcal{B}_1 = \{v_1, \dots, v_n\}$

$\mathcal{B}_2 = \{w_1, \dots, w_n\}$

"matrice di cambiamento di base <u>da</u> $\mathcal{B}_2$ <u>a</u> $\mathcal{B}_1$"

$$
B = \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{n1} & b_{n2} & & b_{nn} \end{pmatrix}
$$

$$
v_i = b_{1i} w_1 + b_{2i} w_2 + \dots + b_{ni} w_n .
$$

<u>Es</u>: $V \supset \mathcal{B}_1 = \{v_1, v_2, v_3\}$, $\mathcal{B}_2 = \{w_1, w_2, w_3\}$ dove

$$
v_1 = w_1 - w_2 + w_3
$$

$$
v_2 = w_2 - w_3
$$

$$
v_3 = w_2 + w_3
$$

La matrice di cambiamento di base da $\mathcal{B}_2$ a $\mathcal{B}_1$ è

$$
B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{pmatrix}
$$

Es:

$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, "f"] \arrow[d, "F_{\mathcal{B}_1}"] & \mathbb{R}^2 \arrow[d, "f_e"] \\
\mathbb{R}^3 \arrow[r, "A_1"'] & \mathbb{R}^2
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "g"] \arrow[d, "F_{\mathcal{B}_2}"] & \mathbb{R}^2 \arrow[d, "F_e"] \\
\mathbb{R}^2 \arrow[r, "A_2"'] & \mathbb{R}^2
\end{tikzcd}
$$

Dati: $\mathcal{B}_1$, $\mathcal{B}_2$, A_1, A_2. Determinare la matrice che rappresenta $g \circ f$ nelle basi canoniche.

$$
\begin{tikzcd}
\mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_e"'] & \mathbb{R}^3 \arrow[r, "f"] \arrow[d, "F_{\mathcal{B}_1}"] & \mathbb{R}^2 \arrow[r, equal] \arrow[d, "f_e"] & \mathbb{R}^2 \arrow[r, "g"] \arrow[d, "F_{\mathcal{B}_2}"] & \mathbb{R}^2 \arrow[d, "F_e"] \\
\mathbb{R}^3 \arrow[rrrr, bend right, "C"'] & \mathbb{R}^3 \arrow[l, "B_1"] \arrow[r, "A_1"'] & \mathbb{R}^2 & \mathbb{R}^2 \arrow[l, "B_2"] \arrow[r, "A_2"'] & \mathbb{R}^2
\end{tikzcd}
$$

La matrice cercata è

$$C = A_2 B_2^{-1} A_1 B_1^{-1}$$

$$\begin{tikzcd} \mathcal{L} : V \arrow[r] \arrow[d, "B_V"'] & W \arrow[d, "B_W"] \\ \mathbb{K}^n \arrow[r] & \mathbb{K}^m \end{tikzcd}$$

$$(e_1 | \dots | e_r | 0 | \dots | 0) = \left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & 0 \end{array} \right)$$

$\mathcal{L}, \mathcal{L}' : V \longrightarrow W$.

$$\mathcal{L} \sim \mathcal{L}' \iff \mathrm{rg}(\mathcal{L}) = \mathrm{rg}(\mathcal{L}')$$

$\underline{S_A}$: $A \in Mat_{2\times 2}(\mathbb{K})$

$$S_A : \mathbb{K}^2 \longrightarrow \mathbb{K}^2$$

$$X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto AX = x_1 A^1 + x_2 A^2$$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad S_A\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} x_1 + 2x_2 \\ 3x_1 + 4x_2 \end{pmatrix}$$

$$\underset{2\times \underline{3}}{A = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 5 \end{pmatrix}} \qquad S_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$$

S

$$S_A\left(\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}\right) = x_1 \begin{pmatrix} 1 \\ 3 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 5 \end{pmatrix}$$

$$= \begin{pmatrix} x_1 + 2x_2 - x_3 \\ 3x_1 + 4x_2 + 5x_3 \end{pmatrix} = \begin{pmatrix} A_1 X \\ A_2 X \end{pmatrix}$$

$$(1, 2, -1)\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 1x_1 + 2x_2 - x_3$$

$$A = \left(\begin{array}{cc|c} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \hline \cdot & \cdot & \cdot \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right)$$

$$A^2 = \left(\begin{array}{c|c} \overset{2}{\overline{A}} & \overset{1}{\overline{B}} \\ \hline C & D \end{array}\right) \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) = \left(\begin{array}{c|c} AA + BC & AB+BD \\ \hline CA + DC & CB+DD \end{array}\right)$$

$$C = (0, 0)$$

$$\begin{pmatrix} a \\ b \end{pmatrix} \begin{pmatrix} x & y \end{pmatrix} = \begin{pmatrix} ax & ay \\ bx & by \end{pmatrix}$$

$$A^2 = \left(\begin{array}{c|c} AA & AB+BD \\ \hline 0 & DD \end{array}\right)$$

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = xa + yb$$

$$\begin{pmatrix} x \\ y \end{pmatrix} 2 = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$$

$$2 \times \cancel{1} \quad \cancel{1} \times 1$$

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \in Mat_{3\times 3}(\mathbb{R}).$$

$\mathcal{B} = \{ B^1, B^2, B^3 \}$ sono lin. Ind.

e quindi formano una base di $\mathbb{R}^3$.

Il Teorema di Classificazione:

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, "S_B"] \arrow[d, "Id_{\mathbb{R}^3} = F_e = F_{\mathcal{B}_1}"'] & \mathbb{R}^3 \arrow[d, "F_{\mathcal{B}} = S_B^{-1}"] \\ \mathbb{R}^3 \arrow[r, "\mathbb{1}_3"'] & \mathbb{R}^3 \end{tikzcd}$$

$F_{\mathcal{B}} = S_B^{-1}$

Se $\mathcal{B}_2 = \mathcal{B}$

Teorema della dimensione:

$$\mathcal{L}: V \longrightarrow W \quad \text{lineare}$$

per trovare una base di $\operatorname{Im} \mathcal{L}$:

1) Troviamo una base $\{v_1, \dots, v_k\}$ di $\operatorname{Ker} \mathcal{L}$

2) Estendiamo questa base ad una base
$$\{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$$
di V

3) Una base di $\operatorname{Im} \mathcal{L}$ è
$$\{\mathcal{L}(v_{k+1}), \dots, \mathcal{L}(v_n)\}.$$

$\Rightarrow \quad \dim \operatorname{Im} \mathcal{L} + \dim \operatorname{Ker} \mathcal{L} = \dim V.$

Es: $A = \begin{pmatrix} 1 & i \\ 1 & i \end{pmatrix}$ $\quad \operatorname{Ker} A = \left\langle \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\rangle$ $\quad S_A : \underset{V}{\mathbb{C}^2} \longrightarrow \underset{W}{\mathbb{C}^2}$

$\mathcal{B}_V = \left\{ \underset{v_1}{\begin{pmatrix} -i \\ 1 \end{pmatrix}}, \underset{v_2}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \right\} \rightsquigarrow \mathcal{B}_{\operatorname{Im} A} = \left\{ A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = A^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$

$\rightsquigarrow \mathcal{B}_W = \left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{w_2}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \right\}$ $\quad w_1 := A v_2$

La matrice che rappresenta A (ovvero S_A) nelle basi $\mathcal{B}_V$ in partenza e $\mathcal{B}_W$ in arrivo è

$$C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} .$$

$$A v_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \underline{0}\, w_1 + \underline{0}\, w_2$$

$$A v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \underline{1}\, w_1 + \underline{0}\, w_2$$

$$A \sim \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Per ottenere $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ dobbiamo invertire l'ordine degli elementi della base $\mathcal{B}_V$.

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}.$$

Es: $\quad V = \mathbb{R}[x]_{\le 2} \qquad W = \mathbb{R}[x]_{\le 1}$

Sia $\mathcal{B}_V = \{ v_1 = 1+x,\ v_2 = 1-x,\ v_3 = 1+x+x^2 \}$ e $\mathcal{B}_W = \{ w_1 = 1-x,\ w_2 = 1+x \}$

Sia $\mathcal{L} : V \to W$ l'unica f.ne lineare t.c.

$$\mathcal{L}(v_1) = 1 + 2x$$
$$\mathcal{L}(v_2) = x$$
$$\mathcal{L}(v_3) = 2 + 3x$$

Trovare la matrice che rappresenta $\mathcal{L}$ nella base canonica in partenza e nella base $\mathcal{B}_W$ in arrivo.

Sol.:

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, "F_e"'] & V \arrow[r, "\mathcal{L}"] \arrow[d, "F_{\mathcal{B}_V}"'] & W \arrow[r, equal] \arrow[d, "F_e"] & W \arrow[d, "F_{\mathcal{B}_W}"] \\
\mathbb{R}^3 \arrow[rrr, bend right, "C"'] & \mathbb{R}^3 \arrow[l] \arrow[r] & \mathbb{R}^2 & \mathbb{R}^2 \arrow[l]
\end{tikzcd}$$

$$B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = B_2$$

La matrice cercata

$$C = B_2^{-1} A B_1^{-1}$$

$$S_C =$$

$$B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = B_2$$

$$C = B_2^{-1} A B_1^{-1}$$