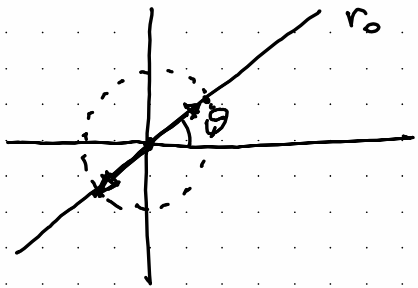
Coseni direttori :

$r: \ ax+by=c \implies$ I coseni direttori sono $\pm \frac{1}{\sqrt{a^2+b^2}}\begin{pmatrix}-b\\ a\end{pmatrix}$

Viceversa : se $v = \begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$ è un versore direttore di r allora

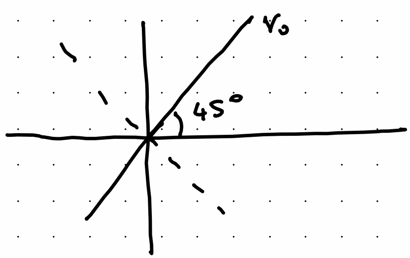
$r_0: \ -\sin\theta \, x + \cos\theta \, y = 0$ giacitura



Es : $v = \begin{pmatrix}\sqrt{2}/2\\ \sqrt{2}/2\end{pmatrix} = \begin{pmatrix}\cos\frac{\pi}{4}\\ \sin\frac{\pi}{4}\end{pmatrix} \ \langle v \rangle =$ la bisettrice del 1° quadrante

$r_0 : \ -\frac{\sqrt{2}}{2} x + \frac{\sqrt{2}}{2} y = 0$

$c=0 \qquad y = x$



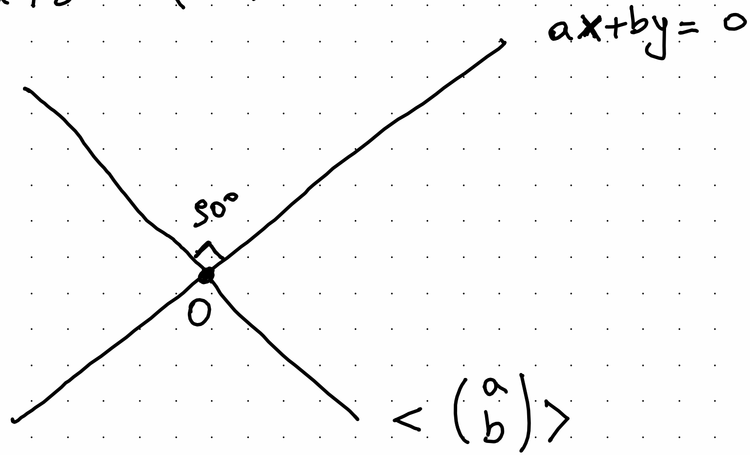
$r: \ ax+by=c$ Un vettore direttore di r è $\begin{pmatrix} -b \\ a \end{pmatrix}$.

Qual'è un vettore ortogonale a $\begin{pmatrix} -b \\ a \end{pmatrix}$? $n = \begin{pmatrix} a \\ b \end{pmatrix}$

$n = \begin{pmatrix} a \\ b \end{pmatrix}$ è un vettore <u>normale</u> a r.

versori normali:

$$\pm \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a \\ b \end{pmatrix}.$$



Trovare una base ortogonale di $(\mathbb{R}^3, b_A)$ dove

·) $A = A^t$

·) $\operatorname{Ker} A = \{0_{\mathbb{R}^3}\}$.

<u>Sol.</u>: (Dim. dell' esistenza di una base ortogonale).

Sia $v_1 \in \mathbb{R}^3$ t.c. $v_1^2 = b_A(v_1, v_1) \neq 0$

Per trovarlo: se $e_1^2 = e_2^2 = e_3^2 = 0$, $e_1 + e_2$, $e_1 + e_3$, $e_2 + e_3$

è non-isotropo. $\Rightarrow$ $\langle v_1 \rangle^{\perp} \ni v_2$ non-isotropo.

<u>Es</u>:

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$b_A(e_i, e_j) = a_{ij} = 0 \quad i \neq j$.

$\Rightarrow \{e_1, e_2, e_3\}$ è ortogonale.

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

$b_A(e_1, e_2) = a_{12} = 1$

$b_A(e_1 + e_2, e_1 + e_2) = e_1^2 + e_2^2 + 2\, b_A(e_1, e_2) = 2 \neq 0$

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad \det A = \det \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} = -\det \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = 2 \neq 0$$

Troviamo una base ortogonale di $(\mathbb{R}^3, b_A)$.

$v_1 = e_1 + e_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ non è isotropo : $(1,1,0) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = (1,1,0) \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 2$

$$\langle v_1 \rangle^{\perp} = \{ X \in \mathbb{R}^3 \mid b_A(v_1, X) = 0 \}$$

a

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; (1,1,0)\, A X = 0 \right\}$$

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; (1,1,2) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \right\}$$

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; x_1 + x_2 + 2x_3 = 0 \right\} = \left\langle \underset{\displaystyle v_2}{\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}}, \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$v_2 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} \qquad v_2^2 = b_A(-e_1 + e_2, -e_1 + e_2) = e_1^2 + e_2^2 - 2\, b_A(e_1, e_2) = -2 \neq 0$

$$\langle v_2 \rangle^{\perp} = \left\{ X \;\middle|\; (-1,1,0) \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} X = 0 \right\} = \{ X \mid (1,-1,0) X = 0 \} = \{ X \mid x_1 - x_2 = 0 \}$$

$$= \left\langle v_1, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle.$$

$v_1+e_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\qquad$ $b_A(v_1+e_3,\, v_1+e_3) = v_1^2 + e_3^2 + 2\, b_A(v_1, e_3)$

$$= 2 + 2\,(1,1,0)\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix}$$

$$= 2 + 2\,(1,1,0)\begin{pmatrix}1\\1\\0\end{pmatrix} = 6 \neq 0$$

$v_3 = v_1+e_3$ non è isotropo. $v_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ non è ortogonale

<u>NO!</u>: <u>$\{v_1, v_2, v_3\}$ è una base ortogonale</u> di $(\mathbb{R}^3, b_A)$:

$$\langle v_1\rangle^\perp \cap \langle v_2^\perp\rangle : \quad \begin{cases} x_1 + x_2 + 2x_3 = 0 \\ x_1 - x_2 = 0 \end{cases}$$

$$\begin{pmatrix}1&1&2\\1&-1&0\end{pmatrix} \sim \begin{pmatrix}1&1&2\\0&-2&-2\end{pmatrix} \sim \begin{pmatrix}1&1&2\\0&1&1\end{pmatrix} \sim \begin{pmatrix}1&0&1\\0&1&1\end{pmatrix}$$

$$\langle v_1\rangle^\perp \cap \langle v_2\rangle^\perp = \left\langle \begin{pmatrix}-1\\-1\\1\end{pmatrix}\right\rangle \Rightarrow v_3 = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$$

$v_1^2 = 2 > 0$, $v_2^2 = -2 < 0$, $v_3^2 = (-1,-1,1)\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}\begin{pmatrix}-1\\-1\\1\end{pmatrix} = (-1,-1,1)\begin{pmatrix}0\\0\\-2\end{pmatrix} = \underset{\overset{\wedge}{0}}{-2}$

$sg(A) = (1, 2)$.

Es:
$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

Calcolare la segnatura di A.

Sol.:

$$\det A = \det \begin{pmatrix} 3 & 0 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} = \det \begin{pmatrix} 3 & 1 \\ 2 & -1 \end{pmatrix} = -3 -2 = -5 \neq 0$$

$\Rightarrow$ b_A è non-degenere $\left(\operatorname{Ker} b_A = \operatorname{Ker} A = \{ 0_{\mathbb{R}^3} \} \right)$.

$e_1^2 = a_{11} = 1 \neq 0$ $\Rightarrow$ $v_1 = e_1$ non è isotropo.

$$\langle e_1 \rangle^\perp = \{ X \mid e_1^t A X = 0 \} = \{ X \mid (1,2,1) X = 0 \} =$$
$$= \{ X \mid x_1 + 2x_2 + x_3 = 0 \} = \left\langle \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}^2 = (-2,1,0) \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} = (-2,1,0) \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix} = -4 \neq 0$$

$$v_2 := \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

$$\langle v_2 \rangle^{\perp} = \{ X \in \mathbb{R}^3 \mid \overbrace{v_2^t A X}^{b_A(v_2, X)} = 0 \}$$

$$= \left\{ X \in \mathbb{R}^3 \mid (-2,1,0) \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} X = 0 \right\}$$

$$= \{ X \in \mathbb{R}^3 \mid (0, -4, -3)\, X = 0 \}$$

$$= \left\{ X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mid -4x_2 - 3x_3 = 0 \right\} = \left\langle e_1, \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix} \right\rangle$$

$$\langle v_1 \rangle^{\perp} \cap \langle v_2 \rangle^{\perp} : \begin{cases} x_1 + 2x_2 + x_3 = 0 \\ 4x_2 + 3x_3 = 0 \end{cases}$$

$$\begin{pmatrix} 1 & 2 & 1 \\ 0 & 4 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 3/4 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1/2 \\ 0 & 1 & 3/4 \end{pmatrix}$$

$$\langle v_1 \rangle^{\perp} \cap \langle v_2 \rangle^{\perp} = \left\langle \underbrace{\begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix}}_{v_3} \right\rangle \qquad \boxed{sg(A) = (2, 1)}$$

$\Rightarrow \{ v_1, v_2, v_3 \}$ è una base ortogonale di $(\mathbb{R}^3, b_A)$

$$v_3^2 = (2,-3,4) \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} = (2,-3,4) \begin{pmatrix} 0 \\ 0 \\ 5 \end{pmatrix} = 20 > 0$$

Calcoliamo una base di Sylvester per

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}$$

$\mathcal{B} = \left\{ v_1 = e_1,\ v_2 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix},\ v_3 = \begin{pmatrix} 2 \\ -3 \\ 4 \end{pmatrix} \right\}$ è una base ortogonale di $(\mathbb{R}^3, b_A)$. $v_1^2 = 1$, $v_2^2 = -4$, $v_3^2 = 20$.

In questa base, b_A si scrive come

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 20 \end{pmatrix}$$

$\sqrt{v_1^2} = 1$, $\sqrt{-v_2^2} = \sqrt{-(-4)} = \sqrt{4} = 2$, $\sqrt{v_3^2} = \sqrt{20}$

Una base di Sylvester è

$$\mathcal{S} = \left\{ E_1 = \frac{v_1}{\sqrt{v_1^2}} = e_1,\ E_2 = \frac{v_3}{\sqrt{v_3^2}} = \begin{pmatrix} 2/\sqrt{20} \\ -3/\sqrt{20} \\ 4/\sqrt{20} \end{pmatrix},\ E_3 = \frac{v_2}{\sqrt{-v_2^2}} = \begin{pmatrix} -1 \\ 1/2 \\ 0 \end{pmatrix} \right\}$$

In queste base f, b_A si scrive

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$$v_1^2 = \lambda > 0 \qquad E_1 = \frac{v_1}{\sqrt{\lambda}}$$

$$E_1^2 = \left(\frac{1}{\sqrt{\lambda}}\right)^2 v_1^2 = \frac{1}{\lambda}\lambda = 1$$

$$v_1^2 = \mu < 0 \qquad E_1 = \frac{v_1}{\sqrt{-\mu}}$$

$$\Rightarrow E_1^2 = \frac{1}{-\mu} v_1^2 = \frac{\mu}{-\mu} = -1$$

Domanda: $V = \operatorname{Ker} b \oplus U$.

$$b|_U : U \times U \longrightarrow \mathbb{K} \quad : \quad b|_U(u_1, u_2) = b(u_1, u_2)$$

OSS: $\operatorname{Ker} b|_U = \{ u \in U \mid b|_U(u, u') = 0 \ \forall u' \in U \}$

$$= \{ u \in U \mid b(u, u') = 0 \ \forall u' \in U \}$$

$$= \{ u \in U \mid b(u, v) = 0 \ \forall v \in V \}$$

$$= \operatorname{Ker} b \cap U = \{ 0_V \}.$$

Se $\mathcal{B}_U = \{ u_1, \dots, u_K \}$ è una base ortogonale di $(U, b|_U)$ e $\mathcal{B}_{\operatorname{Ker} b} \doteq \{ v_{K+1}, \dots, v_n \}$ è una base di $\operatorname{Ker} b$ allora $\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_{\operatorname{Ker} b}$ è una base ortogonale di (V, b):

$b(u_i, u_j) = 0 \quad \forall i \neq j$, $\quad b(u_i, v_j) = 0$ ← $v_j \in \operatorname{Ker} b$

$b(v_i, u_j) = 0 \quad \forall j$ ← $v_i \in \operatorname{Ker} b$.

$v \in (V, b)$ si dice isotropo se $v^2 = 0$.

<u>Es</u>: $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad b = b_A$

$$e_1^2 = a_{11} = 0 = a_{22} = e_2^2$$

$$b_A(e_1+e_2, e_1+e_2) = (1,1)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1,1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \neq 0$$

$$\langle e_1 \rangle^\perp = \{ X \in \mathbb{R}^2 \mid e_1^t A X = 0 \}$$
$$= \{ X \in \mathbb{R}^2 \mid (0,1) X = 0 \} =$$
$$= \{ X \in \mathbb{R}^2 \mid x_2 = 0 \} = \langle e_1 \rangle = U$$

$U \cap U^\perp \neq \{0_V\}$.

$U \cap U^\perp = U$

$\Rightarrow$ e_1 non può far parte di una base ortogonale di $(\mathbb{R}^2, b_A)$.

$\underline{Es}$:
$$\mathcal{B}=\left\{v_1=\begin{pmatrix}1\\1\\1\end{pmatrix}, v_2=\begin{pmatrix}1\\0\\1\end{pmatrix}, v_3=\begin{pmatrix}0\\1\\1\end{pmatrix}\right\}$$

Trovare una base $\mathcal{E}$ di $(\mathbb{R}^3,\cdot)$ t.c. $\mathcal{E}=\{E_1,E_2,E_3\}$

1) $\mathcal{E}$ è ortonormale $\left(\|E_i\|=\sqrt{E_i^2}=1,\ E_i\cdot E_j=0\right)$

2) $\langle v_1\rangle=\langle E_1\rangle$, $\langle v_1,v_2\rangle=\langle E_1,E_2\rangle$

3) $v_i\cdot E_i>0$

$$F_1^2=\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}=1+1+1=3$$

$\underline{Sol.}$:

$$F_1=v_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$F_2=v_2-\frac{v_2\cdot F_1}{F_1^2}F_1=\begin{pmatrix}1\\0\\1\end{pmatrix}-\frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}=\begin{pmatrix}1/3\\-2/3\\1/3\end{pmatrix}=\frac{1}{3}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$$F_3=v_3-\frac{v_3\cdot F_1}{F_1^2}F_1-\frac{v_3\cdot F_2}{F_2^2}F_2=$$

$$=\begin{pmatrix}0\\1\\1\end{pmatrix}-\frac{2}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}-\frac{-1/3}{2/3}\begin{pmatrix}1/3\\-2/3\\1/3\end{pmatrix}=\begin{pmatrix}0\\1\\1\end{pmatrix}-\frac{\cancel{2}\,4}{\cancel{3}\,6}\begin{pmatrix}1\\1\\1\end{pmatrix}+\frac{1}{6}\begin{pmatrix}1\\-2\\1\end{pmatrix}$$

$$=\begin{pmatrix}-1/2\\0\\1/2\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$F_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad F_2 = \frac{1}{3}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad F_3 = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\|F_1\| = \sqrt{3} \qquad \|F_2\| = \frac{1}{3}\sqrt{1+4+1} = \frac{\sqrt{6}}{3} \qquad \|F_3\| = \frac{1}{2}\sqrt{2} = \frac{\sqrt{2}}{2}$$

$$\mathcal{E} = \left\{ E_1 = \frac{F_1}{\|F_1\|} = \begin{pmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1/\sqrt{6} \\ -2/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}, \quad E_3 = \begin{pmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} \right\}$$

è la base con le proprietà richieste.

$$E_1 \cdot v_1 = \frac{3}{\sqrt{3}} > 0 \qquad E_2 \cdot v_2 = \frac{2}{\sqrt{6}} > 0 \qquad E_3 \cdot v_3 = \frac{1}{\sqrt{2}} > 0.$$

$$\|\lambda v\| = |\lambda| \, \|v\|$$

$$\|\lambda v\| = \sqrt{(\lambda v, \lambda v)} = \sqrt{\lambda^2 v^2} = |\lambda| \sqrt{v^2} = |\lambda| \, \|v\|$$

Teo di der. ort.

$U \supseteq \mathcal{B}_U = \{v_1, \dots, v_k\}$ base di U

Ipotesi: $b|_U$ è non-degenere

$$F : V \longrightarrow \mathbb{K}^k$$
$$v \longmapsto \begin{pmatrix} (v, v_1) \\ \vdots \\ (v, v_k) \end{pmatrix}$$

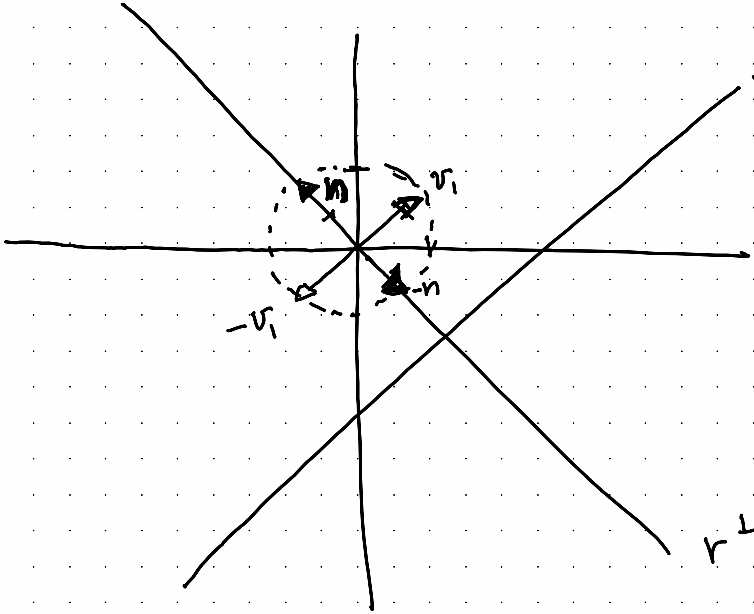
$\{F(v_1), \dots, F(v_k)\}$ è lin. ind. $\Rightarrow$ è una base di $\mathbb{K}^k$

$\Rightarrow$ F è suriettiva $\Rightarrow$ $\dim \operatorname{Ker} F = \dim U^\perp = n - \dim U$.

$n = \dim V$.

$\dim V = \dim U + \dim U^\perp$, $U \cap U^\perp = \{0_V\}$ per ipotesi.

$\Rightarrow$ $V = U \oplus U^\perp = U \oplus \operatorname{Ker} F$



$r: \; ax + by = c$

$$n = \pm \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} a \\ b \end{pmatrix}$$

$$v_1 = \pm \frac{1}{\sqrt{a^2+b^2}} \begin{pmatrix} -b \\ a \end{pmatrix}.$$

$r: \quad 2x+3y=4$

$$r = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, 2x+3y=4 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x+\frac{3}{2}y=2 \right\}$$

$$= \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x=2-\frac{3}{2}y \right\}$$

Teorema di struttura dei sistemi lineari

$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \,\middle|\, x=-\frac{3}{2}y \right\}$$

$$= \begin{pmatrix} 2 \\ 0 \end{pmatrix} + \left\langle \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} \right\rangle$$

$$r = X_0 + \langle v \rangle \qquad v = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\operatorname{Ker} v^t = \operatorname{Ker}(a,b) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \; ax+by=0 \right\} = \left\langle \begin{pmatrix} -b \\ a \end{pmatrix} \right\rangle$$

$r: \quad -bx + ay = c \qquad$ dove $\qquad c = (-b, a) X_0$

<u>Es</u>: $\quad r = \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 3 \\ 7 \end{pmatrix} \right\rangle$

$$\operatorname{Ker}(3,7) = \left\langle \begin{pmatrix} -7 \\ 3 \end{pmatrix} \right\rangle$$

$$r: \; -7x + 3y = c \quad = (-7, 3)\begin{pmatrix} 1 \\ 2 \end{pmatrix} = -7+6 = -1$$

$$\Rightarrow \; r: \; -7x+3y = -1 \qquad \Rightarrow \; r: \; 7x - 3y = 1$$