•) Es 3 sett 11 con MATLAB (Diagonalizzazione)

•) $\mathcal{L}: V \longrightarrow V$ endomorfismo lineare.
Diciamo che un sottospazio vettoriale $U \subseteq V$ è $\mathcal{L}$-invariante se $\mathcal{L}(U) \subseteq U$ ovvero se $\forall u \in U \quad \mathcal{L}(u) \in U$.

•) Teorema spettrale:

$A = A^t$, $\lambda \in Sp(A) \subset \mathbb{R}$

$\{v_1, \dots, v_k\}$ base $V_\lambda(A) = \mathrm{Ker}(\lambda \mathbb{1}_n - A)$

$\rightarrow \{v_1, \dots, v_k, v_{k+1}, \dots, v_n\}$ base di $\mathbb{R}^n$.

$W = \langle v_{k+1}, \dots, v_n \rangle = V_\lambda(A)^\perp$, $\mathbb{R}^n = V_\lambda(A) \overset{\perp}{\oplus} W$.

W è A-invariante: $w \in W \quad Aw \in W$: $v \in V_\lambda(A)$

$$v \cdot Aw = Av \cdot w = \lambda v \cdot w = \lambda (v \cdot w) = 0$$

$w \in V_\lambda(A)^\perp = W$.

Gram-Schmidt

$$\mathcal{B} = \left\{ v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right\} \subset \mathbb{R}^4$$

Applichiamo l'algoritmo di G-S a $\mathcal{B}$:

$$F_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$F_2 = v_2 - pr_{F_1}(v_2) = v_2 - \frac{v_2 \cdot F_1}{F_1 \cdot F_1} v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$F_3 = v_3 - pr_{\langle F_1, F_2 \rangle}(v_3) = v_3 - \frac{v_3 \cdot F_1}{F_1 \cdot F_1} F_1 - \frac{v_3 \cdot F_2}{F_2 \cdot F_2} F_2$$

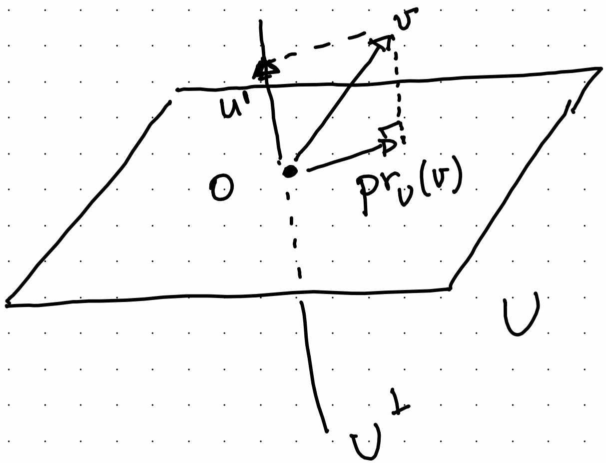
$$= \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{1} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E_1 = \frac{1}{\|F_1\|} F_1, \quad E_2 = \frac{1}{\|F_2\|} F_2, \quad E_3 = \frac{1}{\|F_3\|} F_3$$

$U \subset \mathbb{R}^n$ sottosp. vettoriale.

$$\mathbb{R}^n = U \overset{\perp}{\oplus} U^\perp$$

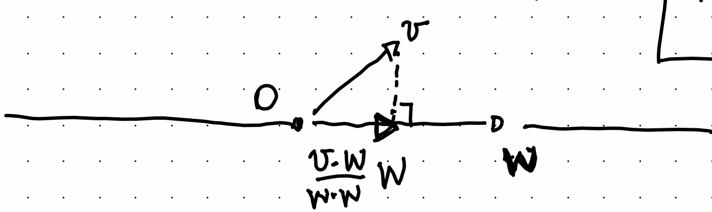
$$\begin{aligned} pr_U : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ v = u + u' &\longmapsto u \end{aligned} \qquad u \in U,\ u' \in U^\perp.$$



$$\boxed{dist(P, U) = \|P - pr_U(P)\|}$$

$U = \langle w \rangle$, $w \neq 0$,

$$\boxed{pr_U(v) = \frac{v \cdot w}{w \cdot w} w}$$



Proiezione ortogonale su sottospazi affini.

$P \in \mathbb{R}^n \qquad U = X_0 + U_0 \subset \mathbb{R}^n$ s.sp. affine.

$$pr_U(P) = X_0 + pr_{U_0}(P - X_0)$$

Es: Trovare la forma canonica metrica e affine della conica $\mathcal{C}_p$ dove

$$p(x,y) = 5x^2+4xy+2y^2-2x+4y-1$$

specificando i cambiamenti di coordinate.

Sol.:

$$p(X) = X^t A X + 2\, b \cdot X + f \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad f = -1$$

$x^2+y^2+\lambda_3 z^2=0$

$sg(A) = ?$

$Tr(A) = \lambda_1+\lambda_2 = 7 > 0$

$det(A) = \lambda_1 \lambda_2 = 6 > 0$

$\Rightarrow \lambda_1, \lambda_2 > 0$

$\Rightarrow sg(A) = (2,0)$.

$$\hat{A} = \left(\begin{array}{c|c} A & b \\ \hline b^t & f \end{array}\right) = \left(\begin{array}{cc|c} 5 & 2 & -1 \\ 2 & 2 & 2 \\ \hline -1 & 2 & -1 \end{array}\right)$$

$$q_{\hat{A}}(X) = X^t \hat{A} X = 5x^2+2y^2-z^2+4xy-2xz+4yz \Big|_{z=1}$$

$$= 5x^2+2y^2-1+4xy-2x+4y = p(X)$$

$$p(X) = X^t A X + 2 b \cdot X + f \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \qquad f = -1$$

1) Diagonalizzare ortogonalmente A:

$$P_A(x) = x^2 - 7x + 6$$

$$x_{1,2} = \frac{7 \pm \sqrt{49-24}}{2} = \frac{7 \pm \sqrt{25}}{2} = \frac{7 \pm 5}{2} = \begin{cases} 6 \\ +1 \end{cases}$$

$$P_A(x) = (x-6)(x-1)$$

Sia λ : $\lambda^2 - 7\lambda + 6 = 0$

$$V_\lambda(A) = \operatorname{Ker}(\lambda \mathbb{1}_2 - A) = \operatorname{Ker}\begin{pmatrix} \lambda-5 & -2 \\ -2 & \lambda-2 \end{pmatrix} =$$

$$= \operatorname{Ker}\begin{pmatrix} 1 & \frac{\lambda-2}{-2} \\ \lambda-5 & -2 \end{pmatrix} = \operatorname{Ker}\begin{pmatrix} 1 & -\frac{1}{2}(\lambda-2) \\ 0 & -2+\frac{1}{2}(\lambda-2)(\lambda-5) \end{pmatrix}$$

$$\mathrm{Ker}\begin{pmatrix} 1 & -\frac{1}{2}(\lambda-2) \\ 0 & -2+\frac{1}{2}(\lambda-2)(\lambda-5) \end{pmatrix} = \mathrm{Ker}\begin{pmatrix} 1 & -\frac{1}{2}(\lambda-2) \\ 0 & -2+\frac{1}{2}4=0 \end{pmatrix}$$

$$(\lambda-2)(\lambda-5) = \lambda^2-7\lambda+10 = -6+10 = 4$$

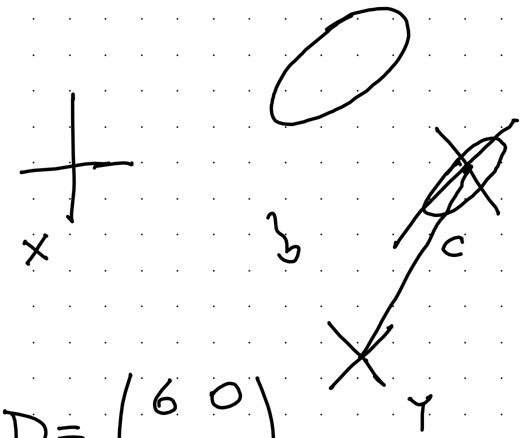
$$= \mathrm{Ker}\left(1, -\frac{(\lambda-2)}{2}\right) = \left\langle \begin{pmatrix} 1 \\ \frac{2}{(\lambda-2)} \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} \lambda-2 \\ 2 \end{pmatrix} \right\rangle$$

$$V_6 = \left\langle \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\rangle \qquad V_1 = \left\langle \begin{pmatrix} -1 \\ 2 \end{pmatrix} \right\rangle$$

$$\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$ è un autovettore di autovalore 6.

$$\begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \Rightarrow \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$ è autovettore di autovalore 1.

$$v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad v_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$



$$E_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} \qquad E_2 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$B = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \Rightarrow B^t A B = D = \begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}$$

Facciamo il cambio di coordinate

① $\boxed{X = BY}$ ovvero $\begin{cases} x_1 = \frac{2}{\sqrt{5}} y_1 - \frac{1}{\sqrt{5}} y_2 \\ x_2 = \frac{1}{\sqrt{5}} y_1 + \frac{2}{\sqrt{5}} y_2 \end{cases}$

$$p(X) = X^t A X + 2 b \cdot X + f = (BY)^t A (BY) + 2b \cdot BY + f$$

$$= Y^t B^t A B Y + 2 B^t b \cdot Y + f = Y^t D Y + 2 B^t b \cdot Y + f$$

$$= q(Y)$$

2) <u>Trasliamo</u> : $Y = Z + C$ dove C serve per ridurre il termine lineare $2 B^t b \cdot Y$

$$q(Y) = Y^t D Y + 2 B^t b \cdot Y + f$$

$$= 6 y_1^2 + y_2^2 + \frac{5}{\sqrt{5}} y_2 - 1$$

$$B^t b = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}^t \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 5 \end{pmatrix}$$

$$q(Y) \underset{Y=Z+C}{=} (Z+C)^t D (Z+C) + 2 B^t b \cdot (Z+C) + f$$

$$= \underline{Z^t D Z} + \underline{Z^t D C} + \underline{C^t D Z} + C^t D C +$$

$$+ 2 \underline{B^t b \cdot Z} + 2 B^t b \cdot C + f$$

$$= Z^t D Z + 2 \left(D C + B^t b \right) \cdot Z + q(C)$$

Cerchiamo C t.c. $DC + B^t b = 0_{\mathbb{R}^2}$:

$$\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} + \frac{1}{\sqrt{5}} \begin{pmatrix} 0 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{cases} 6 c_1 = 0 \\ c_2 = - \dfrac{5}{\sqrt{5}} = -\sqrt{5} \end{cases}$$

$$C := \begin{pmatrix} 0 \\ -\sqrt{5} \end{pmatrix}$$

Se effettuiamo il cambiamento di coordinate

(2) $\boxed{Y = Z + C}$ ovvero $\begin{cases} y_1 = z_1 \\ y_2 = z_2 - \sqrt{5} \end{cases}$

otteniamo

$$p(X) = q(Y) = 6 z_1^2 + z_2^2 + q\left(\begin{pmatrix} 0 \\ -\sqrt{5} \end{pmatrix} \right)$$

$$q(Y) = 6y_1^2 + y_2^2 + \sqrt{5}\, y_2 - 1$$

$$q\left(\begin{pmatrix} 0 \\ -\sqrt{5} \end{pmatrix}\right) = 5 - 5 - 1 = -1$$

$\Rightarrow$ $\mathcal{C}_P$ è metricamente equivalente alla conica

$$6z_1^2 + z_2^2 - 1 = 0$$

ovvero

$$\frac{z_1^2}{\left(\frac{1}{\sqrt{6}}\right)^2} + z_2^2 = 1 \qquad \left(\text{"}\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\text{"}\right)$$

che è un'ellisse reale.

(3) $z_1 = \frac{x}{\sqrt{6}}$ $z_2 = y$ $\Rightarrow$ $\boxed{x^2 + y^2 - 1 = 0}$

forma canonica affine di $\mathcal{C}_P$.

Classificazione metrica:

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 \qquad Sp(\hat{A})$$

$$\lambda_1 x^2 - 2ky \qquad \text{(parabola)}$$

$$\hat{A} = \left(\begin{array}{c|c} A & b \\ \hline b^t & f \end{array}\right) \qquad \hat{A}' = \left(\begin{array}{c|c} A' & b' \\ \hline b'^t & f' \end{array}\right)$$

$$\hat{A} \sim \hat{A}' \overset{def}{\iff} \exists \ \hat{B} = \left(\begin{array}{c|c} B & c \\ \hline 0 & 1 \end{array}\right) \text{ t.c.}$$

$$\hat{B}^t \hat{A} \hat{B} = \hat{A}', \quad B \text{ è invertibile.}$$

$$\iff \quad B^t A B = A'$$

$$rg(A|b) = rg(A'|b')$$

Es:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\det A = \det \begin{pmatrix} 1 & 1 & 2 \\ 0 & -2 & -1 \\ 0 & -1 & -5 \end{pmatrix} = 9 \neq 0$$

$$P_A(x) = \det\left(x \mathbb{1}_3 - A\right) = \det \begin{pmatrix} x-1 & -1 & -2 \\ -1 & x+1 & -1 \\ -2 & -1 & x+1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & -x-1 & 1 \\ x-1 & -1 & -2 \\ -2 & -1 & x+1 \end{pmatrix}$$

$$= \det \begin{pmatrix} 1 & -x-1 & 1 \\ 0 & -1+(x-1)(x+1) & -2-(x-1) \\ 0 & -3-2x & x+3 \end{pmatrix}$$

$$= \det \begin{pmatrix} x^2-2 & -x-1 \\ -3-2x & x+3 \end{pmatrix}$$

$$= \det \begin{pmatrix} x^2-2 & -x-1 \\ -3-2x & x+3 \end{pmatrix}$$

$$= (x^2-2)(x+3) + (x+1)(-3-2x)$$

$$= x^3 + 3x^2 - 2x - 6 - 3x - 2x^2 - 3 - 2x$$

$$= x^3 + x^2 - 7x - 9$$

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$\mathrm{Tr}(A) = -1$$

$$\mathrm{Tr}(A^2) = 6 + 3 + 6 = 15$$

$$\frac{1}{2}(1 - 15) = -7 \quad \checkmark$$

Teorema di Cayley-Hamilton:

$$A^3 + A^2 - 7A - 9\,\mathbb{1}_3 = O_3$$

$$\Rightarrow A\left(A^2 + A - 7\,\mathbb{1}_3\right) = 9\,\mathbb{1}_3 \Rightarrow \boxed{A^{-1} = \frac{1}{9}\left(A^2 + A - 7\,\mathbb{1}_3\right)}$$

$$A^{-1} = \frac{1}{9}\left(A^2 + A - 7\mathbb{1}_3\right)$$

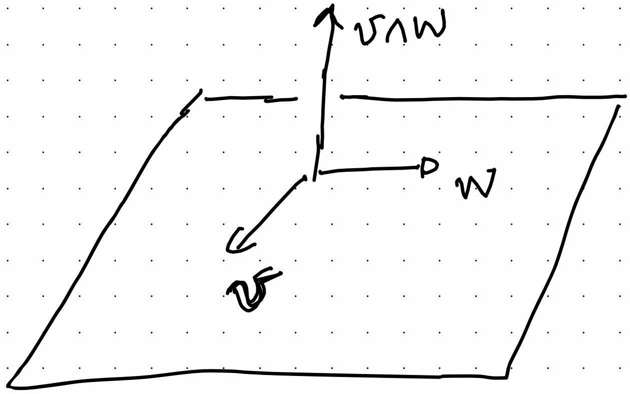
$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 6 \end{pmatrix}$$

$$A^2 + A - 7\mathbb{1}_3 = \begin{pmatrix} 6 & 2 & 1 \\ 2 & 3 & 0 \\ 1 & 0 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} - \begin{pmatrix} 7 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 3 \\ 3 & -5 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$A^{-1} = \frac{1}{9}\begin{pmatrix} 0 & 3 & 3 \\ 3 & -5 & 1 \\ 3 & 1 & -2 \end{pmatrix}$$


v∧w
w
v