Es 1:

$$D = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 2 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \end{pmatrix} \qquad D' = \begin{pmatrix} -1 & \\ & -2 \end{pmatrix}$$

$$DA = \begin{pmatrix} 1 & 1 \\ -2 & -2 \\ 6 & 6 \end{pmatrix} \qquad AD' = \begin{pmatrix} -1 & -2 \\ -2 & -4 \\ -3 & -6 \end{pmatrix}$$

<u>Molt. a sinistra</u>

Se $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ allora $(DA)_i = \lambda_i A_i$

"la i-esima <u>riga</u> di A viene moltiplicata per λ_i".

<u>Molt. a dst</u>:

$$(AD)^j = \lambda_j A^j$$

"la j-esima colonna di A viene moltiplicata per λ_j

Es2: 1) Due matrici $A, B \in Mat_{m\times m}(\mathbb{K})$ sono simili se e solo se $rg(A) = rg(B)$ [Teorema di classificazione]

2) $A \sim B$ se $\exists\ F_1, F_2$

$$\begin{tikzcd} \mathbb{K}^m \arrow[r, "S_A"] \arrow[d, "F_1"', "\simeq"] & \mathbb{K}^m \arrow[d, "F_2", "\simeq"'] \\ \mathbb{K}^m \arrow[r, "S_B"] & \mathbb{K}^m \end{tikzcd} \qquad F_2 \circ S_A = S_B \circ F_1.$$

Def. di simili.

$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad B = \mathbb{1}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad rg\, A = 2$ perché

$\{A^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, A^2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}\}$ sono lin. ind.

$rg\, B = 2$ perché $\{e_1, e_2\}$ sono lin. ind.

Quindi $A \sim B$. Osserviamo che A e B sono invertibili

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "A"] \arrow[d, equal] & \mathbb{C}^2 \arrow[d, "S_{A^{-1}}"] \\ \mathbb{C}^2 \arrow[r, "B = \mathbb{1}_2"] & \mathbb{C}^2 \end{tikzcd}$$

A^{-1}

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "S_A"] \arrow[d, equal] & \mathbb{C}^2 \arrow[d, "S_A^{-1}"] \\ \mathbb{C}^2 \arrow[r, "S_{\mathbb{1}_2}"] & \mathbb{C}^2 \end{tikzcd}$$

$S_A^{-1} = S_{A^{-1}}$: $A^1 = \begin{pmatrix}1\\1\end{pmatrix} \longmapsto e_1$ $\qquad$ S_A : $e_1 \longmapsto A^1 = \begin{pmatrix}1\\1\end{pmatrix}$

$A^2 = \begin{pmatrix}1\\2\end{pmatrix} \longmapsto e_2$ $\qquad$ $e_2 \longmapsto A^2 = \begin{pmatrix}1\\2\end{pmatrix}$

$$e_1 = \begin{pmatrix}2\\2\end{pmatrix} - \begin{pmatrix}1\\2\end{pmatrix} = 2\,A^1 - A^2$$

$$e_2 = -\begin{pmatrix}1\\1\end{pmatrix} + \begin{pmatrix}1\\2\end{pmatrix} = -A^1 + A^2$$

$$S_A^{-1}(e_1) = 2\,S_A^{-1}(A^1) - S_A^{-1}(A^2) = 2e_1 - e_2 = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$S_A^{-1}(e_2) = -S_A^{-1}(A^1) + S_A^{-1}(A^2) = -e_1 + e_2 = \begin{pmatrix}-1\\1\end{pmatrix}$$

$\Rightarrow A^{-1} = \begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}$. Verifica: $A\,A^{-1} = \begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}\begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ ✓

$F_1 = S_{\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}}$ $\qquad$ $F_2 = S_{\begin{pmatrix}2 & -1\\-1 & 1\end{pmatrix}}$.

2) $A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$ $\quad B = \begin{pmatrix} 1+i & 1-i \\ i & 1 \end{pmatrix}$

$\operatorname{rg} A = 2$, $\quad \operatorname{rg} B = 1 \quad (B^1 = i\, B^2)$

$\Rightarrow$ A e B non sono simili.

3) $A = \begin{pmatrix} 1 & 1+i \\ 2 & 2+2i \end{pmatrix}$ $\quad B = \begin{pmatrix} 1+i & 1-i \\ i & 1 \end{pmatrix}$

$\operatorname{rg} A = 1 \quad \operatorname{rg} B = 1 \quad \Rightarrow \quad A \sim B$

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "S_A"] \arrow[d, "S_{\mathcal{B}_1}"'] & \mathbb{C}^2 \arrow[d, "S_{\mathcal{B}_2}"] \\ \mathbb{C}^2 \arrow[r, "S_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}"'] & \mathbb{C}^2 \end{tikzcd}$$

$\operatorname{rg} A = r$

$A \sim \left(\begin{array}{c|c} \mathbb{1}_r & 0 \\ \hline 0 & 0 \end{array}\right)$

$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} 1+i \\ -1 \end{pmatrix}}_{\operatorname{Ker} A} \right\}$ $\quad \mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "S_A"] \arrow[d, "F_{B_1}"'] & \mathbb{C}^2 \arrow[d, "F_{B_2}"] \\ \mathbb{C}^2 \arrow[r, "S_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}"'] & \mathbb{C}^2 \end{tikzcd}$$

$$A = \begin{pmatrix} 1 & 1+i \\ 2 & 2+2i \end{pmatrix}$$

$$B = \begin{pmatrix} 1+i & 1-i \\ i & 1 \end{pmatrix}$$

$$B_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \underset{\text{Ker } A}{\begin{pmatrix} 1+i \\ -1 \end{pmatrix}} \right\} \qquad B_2 = \left\{ \overset{S_A\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}{\overset{\shortparallel}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\begin{tikzcd} \mathbb{C}^2 \arrow[r, "S_B"] \arrow[d, "F_{B_3}"'] & \mathbb{C}^2 \arrow[d, "F_{B_4}"] \\ \mathbb{C}^2 \arrow[r, "S_{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}"'] & \mathbb{C}^2 \end{tikzcd}$$

$$B_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}, \qquad B_4 = \left\{ \overset{S_B\left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right)}{\overset{\shortparallel}{\begin{pmatrix} 1+i \\ i \end{pmatrix}}}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\begin{tikzcd}
\mathbb{C}^2 \arrow[r, "A"] \arrow[d, "F_{\mathcal{B}_1}"'] \arrow[dd, bend right=60, "F_1"'] & \mathbb{C}^2 \arrow[d, "F_{\mathcal{B}_2}"] \arrow[dd, bend left=60, "F_2"] \\
\mathbb{C}^2 \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}"] & \mathbb{C}^2 \\
\mathbb{C}^2 \arrow[u, "F_{\mathcal{B}_3}"] \arrow[r, "B"'] & \mathbb{C}^2 \arrow[u, "F_{\mathcal{B}_4}"']
\end{tikzcd}$$

$$\mathcal{B}_1 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1+i \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B}_2 = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$\mathcal{B}_3 = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -i \end{pmatrix} \right\}$$

$$\mathcal{B}_4 = \left\{ \begin{pmatrix} 1+i \\ i \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

$$F_1 = F_{\mathcal{B}_3}^{-1} \circ F_{\mathcal{B}_1} \quad , \quad F_2 = F_{\mathcal{B}_4}^{-1} \circ F_{\mathcal{B}_2}$$

$F_{\mathcal{B}_1} = S_{B_1}^{-1}$ dove $B_1 = \begin{pmatrix} 1 & 1+i \\ 0 & 1 \end{pmatrix}$

$F_{\mathcal{B}_2} = S_{B_2}^{-1}$ dove $B_2 = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

$F_{\mathcal{B}_3} = S_{B_3}^{-1}$ dove $B_3 = \begin{pmatrix} 1 & 1 \\ 0 & -i \end{pmatrix}$

$F_{\mathcal{B}_4} = S_{B_4}^{-1}$ dove $B_4 = \begin{pmatrix} 1+i & 0 \\ i & 1 \end{pmatrix}$

$$\Rightarrow F_1 = S_{B_3} \circ S_{B_1}^{-1} = S_{B_3 B_1^{-1}}$$

$$F_2 = S_{B_4} \circ S_{B_2}^{-1} = S_{B_4 B_2^{-1}}$$

ES 3 : 1) F $\quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 3 & 1 \end{pmatrix}$ $\quad B = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$ ✓

$3 \times 2 \qquad 2 \times 2$

2) V

3) F $\quad A = \begin{pmatrix} 2 & 0 \\ 2 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 0 \\ 1 & 0 \end{pmatrix} \quad AB = \begin{pmatrix} 4 & 0 \\ 6 & 0 \end{pmatrix} \quad BA = \begin{pmatrix} 4 & 0 \\ 2 & 0 \end{pmatrix}$

4) V

5) F : $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

6) F : $(1, 1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = 0$; $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

7) Falso (v. 3)

8) Falso : $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ (vero se A è invertibile!)

9) FALSO : $A = (1 \ 1) \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad AB = 1$

(vero se A e B sono quadrate!).

$\underline{Es\ 4}$: 1) $A = \underline{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}$ $\quad A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\underbrace{\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}}_{A} \begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$ ottimo!

2) ✓

3) $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ t.c. $AB = BA$ per ogni B.

$$A\,E_{11} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = E_{11} A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad b = 0\ , \ c = 0$$

$$A\,E_{12} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = E_{12} A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = \begin{pmatrix} 0 & d \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \quad a = d.$$

4) V $\quad \begin{pmatrix} a & b \\ & \end{pmatrix}$

$$\begin{pmatrix} 0 & -1/c \\ -1/b & 0 \end{pmatrix}\begin{pmatrix} 0 & -1/c \\ -1/b & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/bc & 0 \\ 0 & 1/bc \end{pmatrix}$$

scegli e $b = -\frac{1}{c}$

$b = 1 \quad b$

$c = -1$

Es 5)

$$T = \left(\begin{array}{cc|c} 1 & 0 & 2 \\ 1 & 0 & 0 \\ \hline 0 & 0 & -1 \end{array}\right) = \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right)$$

$$T^2 = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} = \begin{pmatrix} A^2 & AB+BC \\ 0 & C^2 \end{pmatrix} = \left(\begin{array}{c|c} A & \begin{smallmatrix} 0 \\ 2 \end{smallmatrix} \\ \hline 0 & 1 \end{array}\right)$$

$$A^2 = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} = A$$

$$AB+BC = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 2 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \end{pmatrix}(-1) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$$

$$C^2 = 1$$

$$T^3 = \left(\begin{array}{c|c} A & \begin{smallmatrix} 2 \\ 0 \end{smallmatrix} \\ \hline 0 & -1 \end{array}\right) = T$$

$$T^n = \begin{cases} T & \text{se } n \text{ è dispari} \\ \left(\begin{array}{c|c} A & \begin{smallmatrix} 0 \\ 2 \end{smallmatrix} \\ \hline 0 & 1 \end{array}\right) & \text{se } n \text{ è pari.} \end{cases}$$

$$A = \begin{array}{c} \begin{array}{cc} 2 & 3 \end{array} \\ \left(\begin{array}{c|c} X & Y \\ \hline 0 & Z \end{array}\right) \end{array} \qquad B = \begin{array}{c} 2 \\ 3 \end{array} \left(\begin{array}{c|c} U & 0 \\ \hline V & W \end{array}\right)$$

$$AB = \left(\begin{array}{c|c} XU + YV & YW \\ \hline ZV & ZW \end{array}\right) =$$

3)

$$\begin{pmatrix} 1 & X \\ -Y & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ Y & 1 \end{pmatrix} = \begin{pmatrix} 1+XY & X \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -X \\ 0 & 1 \end{pmatrix} = \mathbb{1}$$

$$\begin{pmatrix} 1 & X \end{pmatrix} \begin{pmatrix} 1 \\ Y \end{pmatrix} = 1 + XY$$

$$\begin{pmatrix} X & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -X \end{pmatrix} = 0$$