Es: $V = \mathbb{R}[x]_{\leq 2}$

$\mathcal{C} = (1, x, x^2)$.

$\mathcal{B} = (1+x\,,\ 1+2x\,,\ 1+x+x^2)$

1) Dimostrare che $\mathcal{B}$ è una base di V

2) Sia $T : V \longrightarrow V$ lineare t.c.

$$T(1+x) = 1$$
$$T(1+2x) = 2$$
$$T(1+x+x^2) = 1+2x^2$$

Scrivere la matrice che rappresenta T nella base $\mathcal{B}$ in partenza e nella base $\mathcal{C}$ in arrivo.

3) Scrivere la matrice che rappresenta T in $\mathcal{C}$

4) $\mathcal{B}_{Ker(T)}$

5) $\mathcal{B}_{Im(T)}$

6) Calcolare $\dim(\operatorname{Ker} T + \operatorname{Im} T)$ e $\operatorname{din}(\operatorname{Ker} T \cap \operatorname{Im} T)$

Sol.: 1) $\dim V = 3$. È quindi sufficiente verificare che $\mathcal{B}$ è lin. Ind., dato che $|\mathcal{B}| = 3 = \dim V$.

Consideriamo $F_{\mathcal{C}}(\mathcal{B}) = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right)$.

Poiché $F_{\mathcal{C}}$ è un isomorfismo lineare, $\mathcal{B}$ è lin. ind. se e solo se $F_{\mathcal{C}}(\mathcal{B})$ lo è.

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow rg(B) = 3 \Rightarrow F_{\mathcal{C}}(\mathcal{B}) \text{ è lin. ind.}$$

2)

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "F_{\mathcal{B}}"'] & V \arrow[d, "F_e"] \\
\mathbb{R}^3 \arrow[r, "S_A"'] & \mathbb{R}^3
\end{tikzcd}$$

$$A = \left(F_e\left(T(1+x)\right) \middle| F_e\left(T(1+2x)\right) \middle| F_e\left(T(1+x+x^2)\right) \right)$$

$$= \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

3)

$$\begin{tikzcd}
V \arrow[r, "T"] \arrow[d, "F_e"'] & V \arrow[d, "F_e"] \\
\mathbb{R}^3 \arrow[r, "S_C"'] & \mathbb{R}^3
\end{tikzcd}$$

$$\begin{tikzcd}
V \arrow[r, equal] \arrow[d, "F_e"'] & V \arrow[r, "T"] \arrow[d, "F_{\mathcal{B}}"] & V \arrow[d, "F_e"] \\
\mathbb{R}^3 \arrow[rr, bend right, "S_c"] & \mathbb{R}^3 \arrow[l, "S_B"] \arrow[r, "S_A"'] & \mathbb{R}^3
\end{tikzcd}$$

$$S_c = S_A \circ S_B^{-1} = S_A \circ S_{B^{-1}}$$

$$B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix} \qquad C = A B^{-1}$$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$C = AB^{-1}$

$\times \; (A|B) \rightsquigarrow (\mathbb{1} \mid \underset{\underset{\underline{\underline{A^{-1}B}}}{\uparrow}}{AB^{-1}})$ $\underline{\underline{\text{FALSO!!}}}$

Calcoliamo B^{-1}

$$(B|\mathbb{1}_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right)$$

$B^{-1} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ verifichiamo ✓

$$C = AB^{-1} = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Quindi:

$T(1) = 0$

$T(x) = 1$

$T(x^2) = 2x^2$

4) $\mathcal{B}_{\ker T} = F_e^{-1}(\mathcal{B}_{\ker C})$

$$C = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\ker C = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 \;\middle|\; \begin{matrix} x_2 = 0 \\ 2x_3 = 0 \end{matrix} \right\} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$$

$$\mathcal{B}_{\ker T} = \left(F_e^{-1}\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right) \right) = (1)$$

5) $\mathcal{B}_{\operatorname{Im} T} = F_e^{-1}(\mathcal{B}_{\operatorname{Im} C}) =$

$$\mathcal{B}_{\operatorname{Im} S_A} = \mathcal{B}_{\operatorname{Col}(C)} = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix} \right)$$

$$\mathcal{B}_{\operatorname{Im} T} = (1, 2x^2)$$

6) Per la formula di Grassmann

$$\dim(\operatorname{Im} T + \ker T) + \dim(\operatorname{Im} T \cap \ker T) =$$

$$= \dim \operatorname{Im} T + \dim \ker T$$

$\operatorname{Im} T \cap \ker T \subset \ker T$ è un s.sp. vett.

$\dim \ker T = 1$, $1 \in \operatorname{Im} T \cap \ker T \Rightarrow \ker T \cap \operatorname{Im} T = \ker T$

$$\dim(\mathrm{Im}\,T + \mathrm{Ker}\,T) = \dim \mathrm{Im}\,T + \dim \mathrm{Ker}\,T - \dim(\mathrm{Ker}\,T \cap \mathrm{Im}\,T)$$

$$= 2 + 1 - 1 = 2$$

$$\Rightarrow \quad \dim(\mathrm{Im}\,T + \mathrm{Ker}\,T) = \dim \mathrm{Im}\,T.$$

$$\Rightarrow \quad \mathrm{Im}\,T + \mathrm{Ker}\,T = \mathrm{Im}\,T.$$

ovvero

$$\mathrm{Ker}\,T \subset \mathrm{Im}\,T.$$

∎

Teorema: Sia $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in Mat_{2\times 2}(\mathbb{K})$.

A è invertibile $\Leftrightarrow$ $ad-bc \neq 0$.

In questo caso

$$A^{-1} = \frac{1}{ad-bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

dim:

Supponiamo $a \neq 0$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \underset{R_1 \mapsto \frac{1}{a} R_1}{\rightsquigarrow} \begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$

$$\underset{R_2 \mapsto R_2 - cR_1}{\rightsquigarrow} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{pmatrix}$$

$$\underset{R_2 \mapsto a R_2}{\rightsquigarrow} \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & ad-bc \end{pmatrix}$$

A invertibile $\Leftrightarrow$ $rg \begin{pmatrix} 1 & \frac{b}{a} \\ 0 & ad-bc \end{pmatrix} = 2$

$\Leftrightarrow$ $ad-bc \neq 0$

In questo caso $(ad-bc \neq 0)$:
calcoliamo A^{-1} con l'algoritmo di
Gauss-Jordan:

$$(A|\mathbb{1}_2) = \left(\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & d-\frac{bc}{a} & -\frac{c}{a} & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & ad-bc & -c & a \end{array}\right)$$

$$\xrightarrow[R_2 \mapsto \frac{1}{ad-bc} R_2]{} \left(\begin{array}{cc|cc} 1 & \frac{b}{a} & \frac{1}{a} & 0 \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$\xrightarrow[R_1 \mapsto R_1 - \frac{b}{a} R_2]{} \left(\begin{array}{cc|cc} 1 & 0 & \overset{\frac{ad}{a(ad-bc)}}{\overset{\parallel}{\boxed{\frac{1}{a} + \frac{bc}{a(ad-bc)}}}} & -\frac{b}{ad-bc} \\ 0 & 1 & -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

Supponiamo $a=0$.

$$A = \begin{pmatrix} 0 & b \\ c & d \end{pmatrix} \underset{R_1 \leftrightarrow R_2}{\leadsto} \begin{pmatrix} c & d \\ 0 & b \end{pmatrix}$$

A è invertibile $\Leftrightarrow$ $\operatorname{rg}\begin{pmatrix} c & d \\ 0 & b \end{pmatrix} = 2$

$$\Leftrightarrow \quad bc \neq 0$$

$$\Leftrightarrow \quad ad-bc \neq 0$$

In questo caso:

$$(A \mid \mathbb{1}_2) = \left(\begin{array}{cc|cc} 0 & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right)$$

$$\leadsto \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & b & 1 & 0 \end{array}\right)$$

$$\underset{R_2 \mapsto \frac{1}{b} R_2}{\leadsto} \left(\begin{array}{cc|cc} c & d & 0 & 1 \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right)$$

$$\underset{R_1 \mapsto R_1 - dR_2}{\leadsto} \left(\begin{array}{cc|cc} c & 0 & -\frac{d}{b} & 1 \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right) \underset{R_1 \mapsto \frac{1}{c} R_1}{\leadsto} \left(\begin{array}{cc|cc} 1 & 0 & -\frac{d}{bc} & \frac{1}{c} \\ 0 & 1 & \frac{1}{b} & 0 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -\frac{d}{bc} & \frac{1}{c} \\ \frac{1}{b} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} -\frac{d}{ad-bc} & -\frac{b}{ad-bc} \\ -\frac{c}{ad-bc} & \frac{a}{ad-bc} \end{pmatrix}$$

$$\Uparrow$$

__Es__ : $V = \mathbb{R}[x]_{\leq 2}$

$$L = Val_{(-1)} : V \longrightarrow \mathbb{R}$$
$$p \longmapsto p(-1)$$

(Es $L(1+x+x^2) = 1$)

1) Trovare la matrice associata ad L nelle basi canoniche $\mathcal{E} = (1, x, x^2)$ di V e $\mathcal{E} = (1)$ di $\mathbb{R}$.

2) $\mathcal{B}_{KerL}$

3) $\mathcal{B}_{ImL}$

__Sol.__ :

$$\begin{tikzcd}
V \arrow[r, "L"] \arrow[d, "F_{\mathcal{E}}"'] & \mathbb{R} \arrow[d, "F_{\mathcal{E}} = Id"] \\
\mathbb{R}^3 \arrow[r, "S_A"'] & \mathbb{R}
\end{tikzcd}$$

$$A = \left(F_{\mathcal{E}}(L(1)) \,\middle|\, F_{\mathcal{E}}(L(x)) \,\middle|\, F_{\mathcal{E}}(L(x^2)) \right)$$

$L(1) = 1$

$L(x) = -1$

$L(x^2) = 1$

$$A = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$$

$$\begin{tikzcd} V \arrow[r, "L"] \arrow[d, "F_e"'] & \mathbb{R} \arrow[d, equal] \\ \mathbb{R}^3 \arrow[r, "S_A"'] & \mathbb{R} \end{tikzcd}$$

$A = (1 \ -1 \ 1)$

$\mathrm{Ker}\, A : \ x_1 - x_2 + x_3 = 0$

$x_1 = x_2 - x_3$

$$\mathcal{B}_{\mathrm{Ker}\, L} = F_e^{-1}(\mathcal{B}_{\mathrm{Ker}\, A})$$

$$\mathcal{B}_{\mathrm{Ker}\, A} = \left(\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\mathcal{B}_{\mathrm{Ker}\, L} = \left(1 + X , -1 + X^2 \right)$$

$$\mathcal{B}_{\mathrm{Im}\, L} = (1)$$

(S_A è suriettiva

$\Rightarrow$ L è suriettiva).

Es (Giugno 2022)

$$V = \mathbb{R}[x]_{\leq 2}, \quad W = \mathbb{R}[x]_{\leq 4}$$

$$F : V \longrightarrow W$$

$$F(p) = p(x^2) - p(x)$$

1) Dimostrare che F è lineare
2) Calcola $F(x^2-1)$
3) Scrivere la matrice associata a F nelle basi standard
4) Calcolare una base del nucleo ed una base dell'immagine di F.

Sol. 1) $F = \mathrm{Val}_{x^2} - \mathrm{Val}_x$

Quindi F è combinazione lineare di f.ni lineari e quindi è lineare.

2) $F(x^2-1) = (x^4-1) - (x^2-1) = x^4 - x^2$

3)

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "F_e"'] & W \arrow[d, "F_e"] \\ \mathbb{R}^3 \arrow[r, "S_A"] & \mathbb{R}^5 \end{tikzcd}$$

$$A = \left(F_e(F(1)) \,\middle|\, F_e(F(x)) \,\middle|\, F_e(F(x^2)) \right)$$

$$\begin{tikzcd} V \arrow[r, "F"] \arrow[d, "F_e"'] & W \arrow[d, "F_e"] \\ \mathbb{R}^3 \arrow[r, "S_A"] & \mathbb{R}^5 \end{tikzcd}$$

$A = \left(F_e(F(1)) \,\middle|\, F_e(F(x)) \,\middle|\, F_e(F(x^2)) \right)$

$$F(p) = p(x^2) - p(x)$$

$F(1) = 0$

$F(x) = x^2 - x$

$F(x^2) = x^4 - x^2$

$$A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

3) $\operatorname{Ker} A = \operatorname{Ker} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} = \operatorname{Ker} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \left\langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle$

$\Rightarrow \mathcal{B}_{\operatorname{Ker} F} = F_e^{-1} \mathcal{B}_{\operatorname{Ker} A} = (1)$

$\mathcal{B}_{\operatorname{Col} A} = \left(\begin{pmatrix} 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right)$

$\Rightarrow \mathcal{B}_{\operatorname{Im} F} = F_e^{-1} \mathcal{B}_{\operatorname{col} A} = \left(-x + x^2, -x^2 + x^4 \right).$

Proiettori o proiezioni

$V = U \oplus W$

$$\begin{aligned} pr_U^W : \ & V \longrightarrow V \\ & u+w \longmapsto u \end{aligned}$$

Sia $\mathcal{B}_U$ una base di U e sia $\mathcal{B}_W$ una base di W $\mathcal{B} = \mathcal{B}_U \cup \mathcal{B}_W$ è una base di V. Scriviamo la matrice associata pr_U^W nella base $\mathcal{B}$.

$$\begin{tikzcd} V \arrow[r, "pr_U^W"] \arrow[d, "F_{\mathcal{B}}"'] & V \arrow[d, "F_{\mathcal{B}}"] \\ \mathbb{K}^n \arrow[r, "S_A"'] & \mathbb{K}^n \end{tikzcd}$$

$$\mathcal{B}_U = (u_1, \dots, u_r)$$

$$\mathcal{B}_W = (w_{r+1}, \dots, w_n)$$

$$\mathcal{B} = (u_1, \dots, u_r, w_{r+1}, \dots, w_n)$$

$$A = \left(F_{\mathcal{B}}(pr_U^W(u_1)) \,\middle|\, F_{\mathcal{B}}(pr_U^W(u_2)) \,\middle|\, \dots \right)$$

$$\begin{array}{rcl}
pr_U^W : \quad V & \longrightarrow & V \\
u_1 & \longmapsto & u_1 \\
u_2 & \longmapsto & u_2 \\
\vdots & & \\
u_r & \longmapsto & u_r \\
w_{r+1} & \longmapsto & 0 \\
\vdots & & \vdots \\
w_n & \longmapsto & 0
\end{array}$$

$$A = (e_1 \mid e_2 \mid \dots \mid e_r \mid 0 \mid \dots \mid 0)$$

$r = 2, \; n = 4$

$$A = \left(e_1 \mid e_2 \mid 0 \mid 0 \right)$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right) = \left(\begin{array}{c|c} \mathbb{1}_2 & 0 \\ \hline 0 & 0 \end{array} \right)$$

$$A = \begin{pmatrix} \mathbb{1}_r & 0 \\ 0 & 0 \end{pmatrix} \qquad \boxed{A^2 = A}$$

Prop.: Sia $f: V \to V$ lineare. Allora f è un proiettore se e solo se $f^2 = f$.

dim: Se f è un proiettore allora $f = pr_U^W$ dove $U = \mathrm{Im} f$, $W = \mathrm{Ker} f$ e $U \oplus W = V$.
Dato $v = u + w \in U \oplus W$,

$$f^2(v) = f(f(v)) = f(u) = u = f(v)$$

$$\Rightarrow f^2 = f.$$

Viceversa se $f^2 = f$, poniamo $U = \mathrm{Im} f$ e $W = \mathrm{Ker} f$. Mostriamo che $V = U \oplus W$ e $f = pr_U^W$.
Sia $v \in \mathrm{Im} f \cap \mathrm{Ker} f$. Allora $\exists\, v' \in V$ t.c.

$v = f(v')$ e $f(v) = 0_V$. Quindi

$$f(v') = f^2(v') \Rightarrow f^2(v') - f(v') = 0$$

$$\Rightarrow f(f(v') - v') = 0$$

$$\Rightarrow f(v - v') = 0$$

$$\Rightarrow f(v) - f(v') = 0$$

$$\Rightarrow v = f(v') = f(v) = 0 \quad -$$

$\Rightarrow \mathrm{Im} f \cap \mathrm{Ker} f = \{0_V\}$. Per la formula della dimensione, $\dim \operatorname{Ker} f + \dim \operatorname{Im} f = \dim V$

Dalla formula di Grassmann, otteniamo

$$\dim(\operatorname{Im} f + \operatorname{Ker} f) = \dim \operatorname{Im} f + \dim \operatorname{Ker} f - \dim(\operatorname{Im} f \cap \operatorname{Ker} f)$$

$$= \dim \operatorname{Im} f + \dim \operatorname{Ker} f = \dim V$$

$$\Rightarrow \quad \operatorname{Im} f + \operatorname{Ker} f = V$$

$$\Rightarrow \quad V = \operatorname{Im} f \oplus \operatorname{Ker} f.$$

$\forall\, v \in V \ \exists!\, u = f(u') \in \operatorname{Im} f$ e $w \in \operatorname{Ker} f$ t.c. $v = u + w$

$$f(v) = f(u+w) = f(u) + f(w) = f^2(u') + 0$$
$$= f(u') = u$$

Quindi

$$f = pr_U^W. \qquad \blacksquare$$