$$p(x) = 2x_1^2 + 2x_1x_2 + 2x_2^2 + 2x_1 - 2x_2 + 2$$

$$= X^t A X + 2\, b \cdot X + 2$$

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad b = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$P_A(x) = x^2 - 4x + 3 \qquad x_{1,2} = \frac{4 \pm \sqrt{16-12}}{2} = 2 \pm 1 = \begin{cases} 3 \\ 1 \end{cases}$$

$$\lambda : \quad \lambda^2 - 4\lambda + 3 = 0$$

$$V_\lambda(A) = \mathrm{Ker}\,(\lambda \mathbb{1}_2 - A) = \mathrm{Ker}\begin{pmatrix} \lambda-2 & -1 \\ -1 & \lambda-2 \end{pmatrix} =$$

$$= \mathrm{Ker}\,(\lambda-2 \quad -1) = \left\langle \begin{pmatrix} 1 \\ \lambda-2 \end{pmatrix} \right\rangle$$

$$V_3(A) = \left\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\rangle, \quad V_1(A) = \left\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\rangle$$

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \qquad D = \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \qquad B^t A B = D$$

$$X = BY$$

$$p(BY) = (BY)^t A (BY) + 2\, b \cdot BY + 2$$

$$= Y^t B^t A B Y + 2\, B^t b \cdot Y + 2$$

$$= Y^t D Y + 2\, B^t b \cdot Y + 2 = q(Y)$$

Cerchiamo $C \in \mathbb{R}^2$ (se esiste) t.c. $Y = Z + C$ la parte lineare "si annulle":

$$q(Z+C) = (Z+C)^t D (Z+C) + 2\, B^t b \cdot (Z+C) + 2 =$$

$$= Z^t D Z + \underline{Z^t D C} + \underline{C^t D Z} + C^t D C +$$
$$+ 2\, \underline{B^t b \cdot Z} + 2\, B^t b \cdot C + 2 =$$

$$= Z^t D Z + 2\,(B^t b + DC) \cdot Z + q(C)$$

Cerchiamo C t.c. $B^t b + DC = 0$

$$B^t b = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix}$$

$$DC = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix} \qquad \begin{cases} 3c_1 = 0 \\ c_2 = -\sqrt{2} \end{cases}$$

$$C = \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix}$$

$$q(C) = C^t D C + 2\, B^t b \cdot C + 2$$

$$= (0 \ \ -\sqrt{2}) \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix} + 2 \begin{pmatrix} 0 \\ \sqrt{2} \end{pmatrix} \cdot \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix} + 2$$

$$= 2 + 2 \cdot (-2) + 2 = 0$$

$$q(Z+C) = Z^t D Z = 3z_1^2 + z_2^2$$

$$\mathcal{C} : \quad 3z_1^2 + z_2^2 = 0$$

$$x = \frac{1}{\sqrt{3}} z_1 \quad y = z_2 \quad : \quad \mathcal{C} : x^2 + y^2 = 0$$

$r_1 = P_1 + \langle v_1 \rangle \qquad r_2 = P_2 + \langle v_2 \rangle$

Cerchiamo $r_3 = P_3 + \langle v_3 \rangle$ t.c.

$\checkmark\ r_3 \perp r_1, \quad r_3 \perp r_2, \checkmark$

$r_3 \cap r_1 \neq \phi, \quad r_3 \cap r_2 \neq \phi.$

$v_3 = v_1 \wedge v_3$

Cerchiamo s_1, s_2, t_1, t_2 t.c.

$$\begin{cases} P_3 + s_1 (v_1 \wedge v_2) = P_1 + t_1 v_1 \\ P_3 + s_2 (v_1 \wedge v_2) = P_2 + t_2 v_2 \end{cases}$$

$$\begin{cases} P_3 = P_1 + t_1 v_1 - s_1 (v_1 \wedge v_2) \\ P_3 = P_2 + t_2 v_2 - s_2 (v_1 \wedge v_2) \end{cases}$$

$$P_1 + t_1 v_1 - s_1 (v_1 \wedge v_2) = P_2 + t_2 v_2 - s_2 (v_1 \wedge v_2)$$

$$P_1 - P_2 = -t_1 v_1 + t_2 v_2 + (s_1 - s_2)(v_1 \wedge v_2)$$

$$(v_1 | v_2 | v_1 \wedge v_2)\, X = P_1 - P_2$$

$$v_1 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \quad v_2 = \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \quad v_1 \wedge v_2 = \begin{pmatrix} 3 \\ 3 \\ 0 \end{pmatrix} \quad P_1 - P_2 = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ -1 & -2 & 3 & 1 \\ 1 & -1 & 0 & -1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 0 & 6 & 2 \\ 0 & -3 & -3 & -2 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & 1 & 1 & 2/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & 0 & 0 & -2/3 \\ 0 & 1 & 0 & 1/3 \\ 0 & 0 & 1 & 1/3 \end{array}\right)$$

$$P_1 - P_2 = -\frac{2}{3} v_1 + \frac{1}{3} v_2 + \frac{1}{3} (v_1 \wedge v_2)$$

$$\Rightarrow \boxed{P_1 + \frac{2}{3} v_1 = P_2 + \frac{1}{3} v_2 + \frac{1}{3} (v_1 \wedge v_2)} \quad \text{Poniamo}$$

$$P_3 = P_2 + \frac{1}{3} v_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} = \begin{pmatrix} 8/3 \\ 1/3 \\ 5/3 \end{pmatrix}$$

Allora

$$r_3 = P_3 + \langle v_1 \wedge v_2 \rangle = \begin{pmatrix} 8/3 \\ 1/3 \\ 5/3 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle : \begin{cases} x - y + z = 4 \\ 2x - 2y - z = 3 \end{cases}$$

(i coefficienti sono v_1 e v_2)

<u>Soluzione 2</u>:

Cerchiamo un piano π_1 t.c.

$$\left.\begin{array}{l} \cdot)\ r_1 \subset \pi_1 \\ \cdot)\ v_1 \wedge v_2 \in (\pi_1)_0 \end{array}\right] \quad (\pi_1)_0 = \langle v_1, v_1 \wedge v_2 \rangle$$

$\pi_1 = P_1 + \langle v_1, v_1 \wedge v_2 \rangle.$

Cerchiamo un piano π_2 t.c.

$$\left.\begin{array}{l} \cdot)\ r_2 \subset \pi_2 \\ \cdot)\ v_1 \wedge v_2 \in (\pi_2)_0 \end{array}\right] \quad (\pi_2)_0 = \langle v_2, v_1 \wedge v_2 \rangle$$

$\pi_2 = P_2 + \langle v_2, v_1 \wedge v_2 \rangle$

$\pi_1 \cap \pi_2 = r_3$ Infatti:

$$r_3 = P_3 + \langle v_1 \wedge v_2 \rangle \quad (r_3 \perp r_1,\ r_3 \perp r_2)$$

$$\pi_1 = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} + \left\langle \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \qquad \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 2 \end{pmatrix}$$

$\pi_1 : \ -x + y + 2z = 1$

$$\pi_2 = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} + \left\langle \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\rangle \qquad \begin{pmatrix} 2 \\ -2 \\ -1 \end{pmatrix} \wedge \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}$$

$\pi_2 : \ x - y + 4z = 9$

$$r_3 : \begin{cases} -x + y + 2z = 1 \\ x - y + 4z = 9 \end{cases}$$

$$\left(\begin{array}{ccc|c} -1 & 1 & 2 & 1 \\ 1 & -1 & 4 & 9 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 4 & 9 \\ -1 & 1 & 2 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 4 & 9 \\ 0 & 0 & 6 & 10 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 4 & 9 \\ 0 & 0 & 1 & 5/3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 7/3 \\ 0 & 0 & 1 & 5/3 \end{array}\right)$$

r_3 di prima: $\begin{cases} x - y + z = 4 \\ 2x - 2y - z = 3 \end{cases}$

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 2 & -2 & -1 & 3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 0 & -3 & -5 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 1 & 4 \\ 0 & 0 & 1 & 5/3 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|c} 1 & -1 & 0 & 7/3 \\ 0 & 0 & 1 & 5/3 \end{array}\right)$$

Concludiamo che la soluzione 1 e 2 danno la stessa retta.

Es 2 :

$U = \langle v_1, v_2 \rangle \qquad W = \langle v_1, v_2 \rangle^{\perp} = \langle v_1 \wedge v_2 \rangle$

$\mathbb{R}^3 = U \oplus W \qquad A \;\; 3\times 3$

$S_A : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$S_A(v_1) = -3v_1 \qquad v_1 \longmapsto -3v_1$

$S_A(v_2) = -3v_2 \qquad v_2 \longmapsto -3v_2$

$S_A(v_1 \wedge v_2) = 5\, v_1 \wedge v_2 \qquad v_1\wedge v_2 \longmapsto 5\, v_1 \wedge v_2$

Nella base $\mathcal{B} = (v_1, v_2, v_1 \wedge v_2)$ la matrice che rappresenta S_A è

$$-3v_1 = -3v_1 + 0v_2 + 0\, v_1\wedge v_2$$
$$-3v_2 = 0v_1 - 3v_2 + 0\, v_1\wedge v_2$$
$$5\, v_1\wedge v_2 = 0v_1 + 0v_2 + 5\, v_1\wedge v_2$$

$$C = \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{pmatrix}$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_e"'] & \mathbb{R}^3 \arrow[r, "S_A"] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[r, equal] \arrow[d, "F_{\mathcal{B}}"] & \mathbb{R}^3 \arrow[d, "F_e"] \\ \mathbb{R}^3 & \mathbb{R}^3 \arrow[l, "S_B"] \arrow[r, "S_C"'] & \mathbb{R}^3 \arrow[r, "S_B"'] & \mathbb{R}^3 \end{tikzcd}$$

$$B = (v_1 | v_2 | v_1\wedge v_2) = \begin{pmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$$

$A = BCB^{-1}$

$$BC = \begin{pmatrix} -3 & -3 & -5 \\ 3 & -3 & -5 \\ 0 & -3 & 10 \end{pmatrix}$$

$B^{-1} =$ con Gauss-Jordan

$$(B | \mathbb{1}_3) = \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ -1 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & -2 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right) \rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 3 & -1/2 & -1/2 & 1 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/6 & -1/6 & 1/3 \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 5/6 & -1/6 & 1/3 \\ 0 & 1 & 0 & 2/6 & 2/6 & 1/3 \\ 0 & 0 & 1 & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{3} \end{array}\right)$$

$$\rightsquigarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/6 & -3/6 & 0 \\ 0 & 1 & 0 & 2/6 & 2/6 & 2/6 \\ 0 & 0 & 1 & -1/6 & -1/6 & 1/3 \end{array}\right)$$

$$B^{-1} = \frac{1}{6}\begin{pmatrix} 3 & -3 & 0 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{pmatrix}$$

$$A = BCB^{-1} = \frac{1}{6}\begin{pmatrix} -3 & -3 & -5 \\ 3 & -3 & -5 \\ 0 & -3 & 10 \end{pmatrix}\begin{pmatrix} 3 & -3 & 0 \\ 2 & 2 & 2 \\ -1 & -1 & 2 \end{pmatrix}$$

$$= \frac{1}{6}\begin{pmatrix} -10 & 8 & -16 \\ 8 & -10 & -16 \\ -16 & -16 & 4 \end{pmatrix}$$

Verifichiamo:

$$Av_1 = \frac{1}{6}\begin{pmatrix} -10 & 8 & -16 \\ 8 & -10 & -16 \\ -16 & -16 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} -18 \\ 18 \\ 0 \end{pmatrix} =$$

$$= -3v_1$$

$Av_2 = 3v_2$

$A(v_1\wedge v_2) = \dots = 5\, v_1\wedge v_2$

$r_1 = P_1 + \langle v_1 \rangle \qquad r_2 = P_2 + \langle v_2 \rangle$

$r_3 = P_3 + \langle v_1 \wedge v_2 \rangle$

$r_3 \cap r_1 \neq \phi \iff rg\,(v_1 \,|\, v_1 \wedge v_2 \,|\, P_3 - P_1) = 2$

$r_3 \cap r_2 \neq \phi \iff rg\,(v_2 \,|\, v_1 \wedge v_2 \,|\, P_3 - P_2) = 2$

$P_3 - P_1 \in \langle v_1, v_1 \wedge v_2 \rangle$

$P_3 - P_2 \in \langle v_2, v_1 \wedge v_2 \rangle$

$$P_3 = P_1 + t_1 v_1 + s_1\, v_1 \wedge v_2$$
$$\parallel$$
$$P_3 = P_2 + t_2 v_2 + s_2\, v_1 \wedge v_2$$

$$P_1 - P_2 = -t_1 v_1 + t_2 v_2 + (s_2 - s_1)\, v_1 \wedge v_2$$

$$(v_1 \,|\, v_2 \,|\, v_1 \wedge v_2)\, X = P_1 - P_2$$

$$\det \begin{pmatrix} 1 & 1 & x-3 \\ -1 & 1 & y-2 \\ 1 & 0 & z-1 \end{pmatrix} = 0 \qquad -x + y + 2z = 1$$

$$\det \begin{pmatrix} 2 & 1 & x-2 \\ -2 & 1 & y-1 \\ -1 & 0 & z-2 \end{pmatrix} = 0 \qquad x - y + 4z = 9$$

Es 4 :

$$\mathbb{R}^4 = U \oplus U^\perp \iff P_U + P_{U^\perp} = \mathbb{1}_4$$

$$X = P_U X + P_{U^\perp} X$$

$$P_{U^\perp} X = X - P_U X$$

$$\boxed{P_{U^\perp} = \mathbb{1}_4 - P_U}$$

$$A e_i = A^i$$

$$e_i = P_U e_i + P_{U^\perp} e_i$$

$$(P_{U^\perp})^i = P_{U^\perp} e_i = e_i - P_U e_i$$
$$= (\mathbb{1}_4)^i - (P_U)^i$$

$$U^\perp = \operatorname{Ker} A^t = \langle w_1, w_2 \rangle$$

$$B = (w_1 | w_2) \qquad P_{U^\perp} = B (B^t B)^{-1} B^t = \mathbb{1}_4 - P_U$$

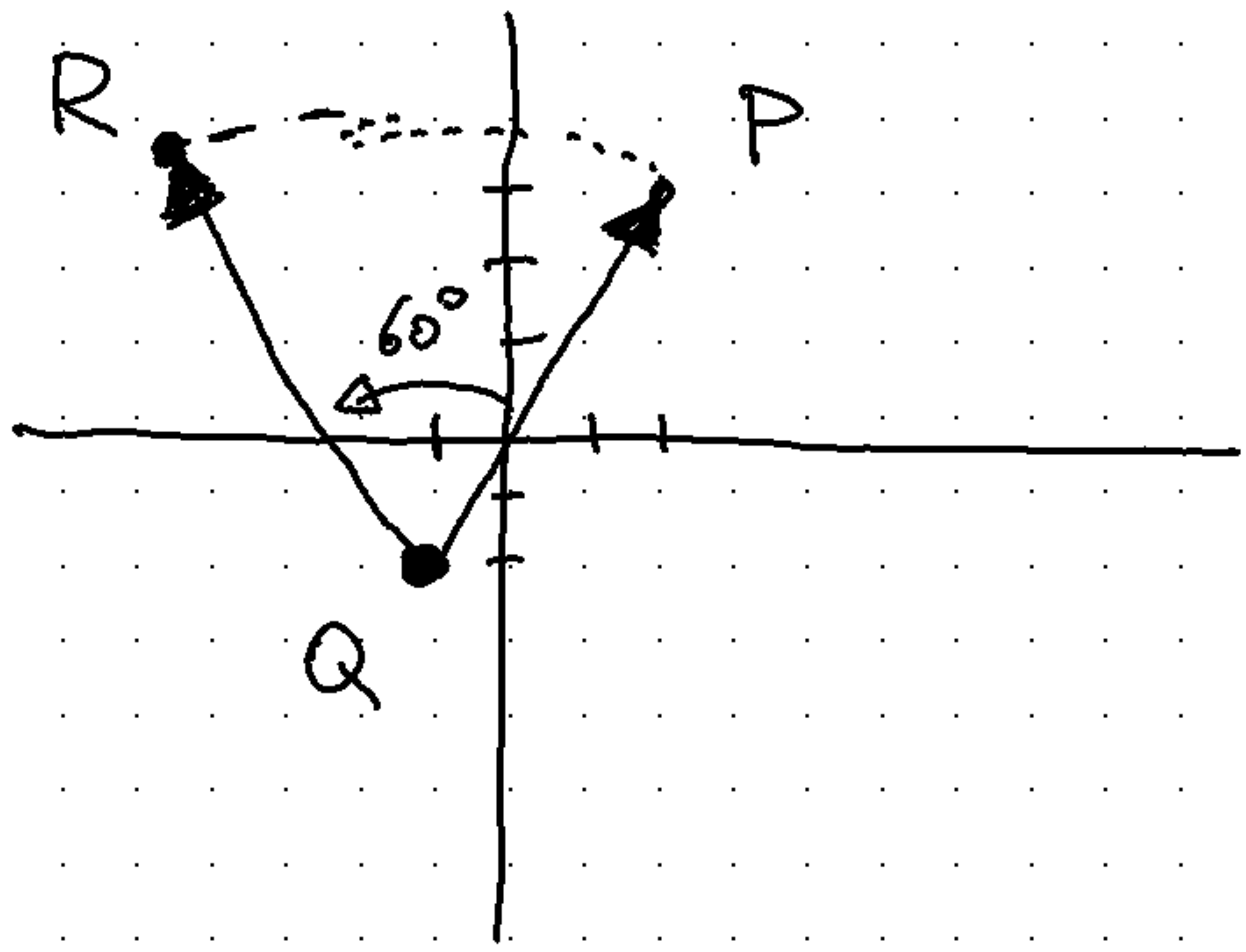
$$F_1 = v_1$$
$$F_2 = v_2 - \frac{v_2 \cdot F_1}{F_1 \cdot F_1} F_1$$

$$pr_U (w) = \frac{w \cdot F_1}{F_1 \cdot F_1} F_1 + \frac{w \cdot F_2}{F_2 \cdot F_2} F_2$$
$$= P_U w$$

$$\operatorname{dist}(w, U) = \| w - pr_U (w) \|$$

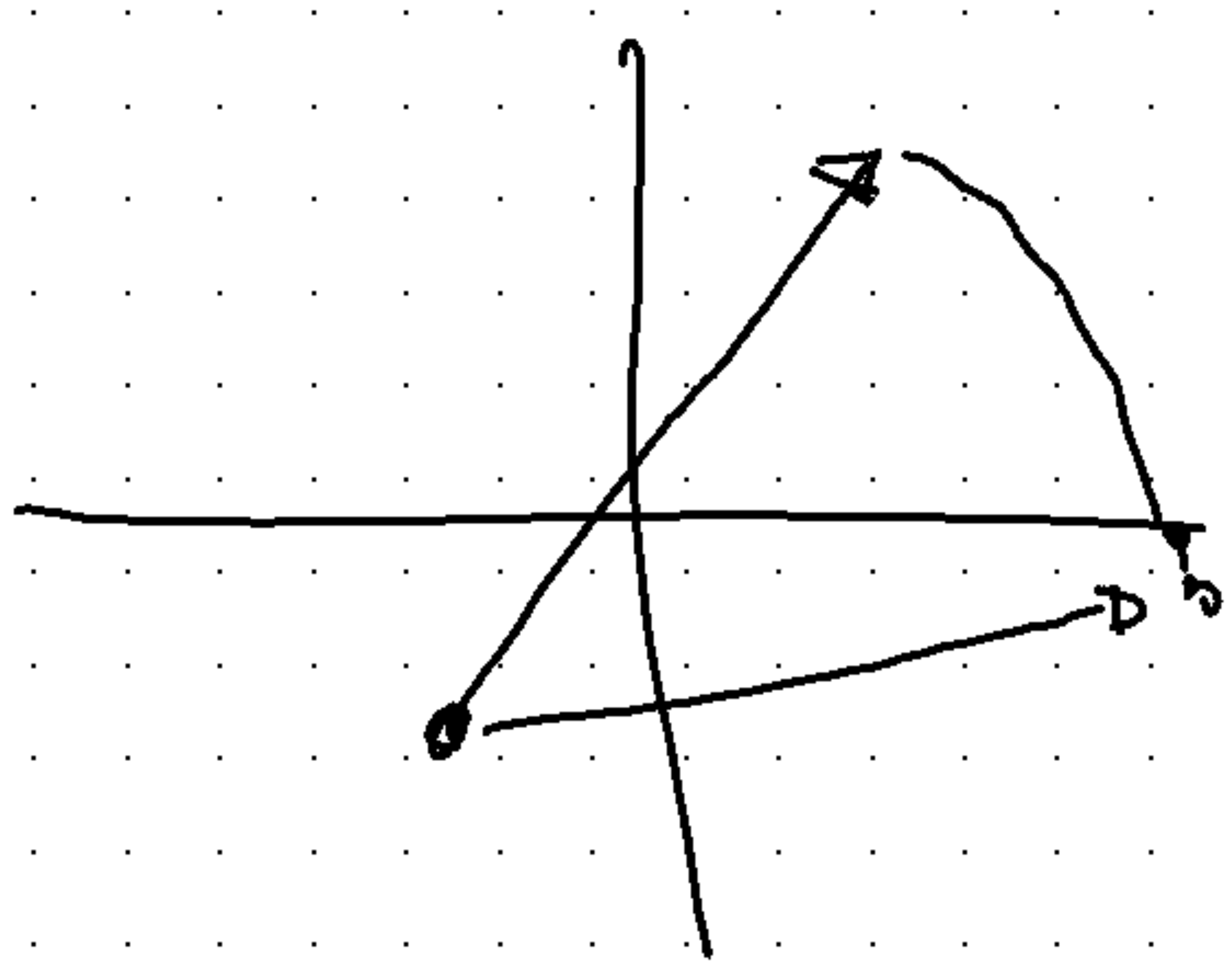
Es: Trovare il punto ottenuto ruotando di $60°$ in senso antiorario il punto $P = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ attorno al punto $\begin{pmatrix} -1 \\ -2 \end{pmatrix} = Q$

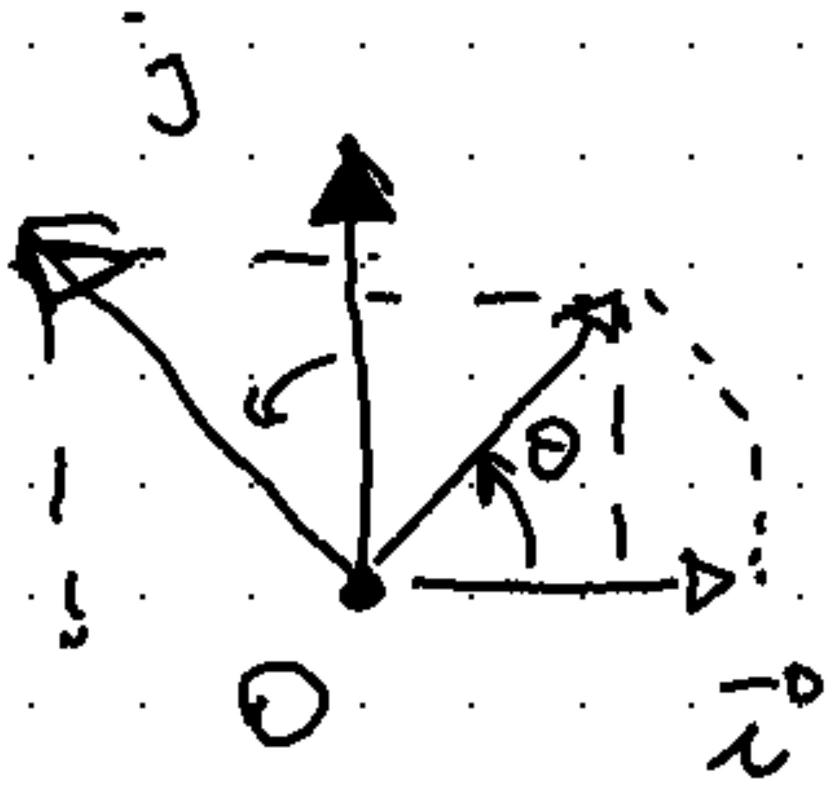
Sol.:



$$R = Q + R_{\frac{\pi}{3}} (P - Q)$$

$$= Q + \begin{pmatrix} \cos \pi/3 & -\sin \pi/3 \\ \sin \pi/3 & \cos \pi/3 \end{pmatrix} (P - Q)$$



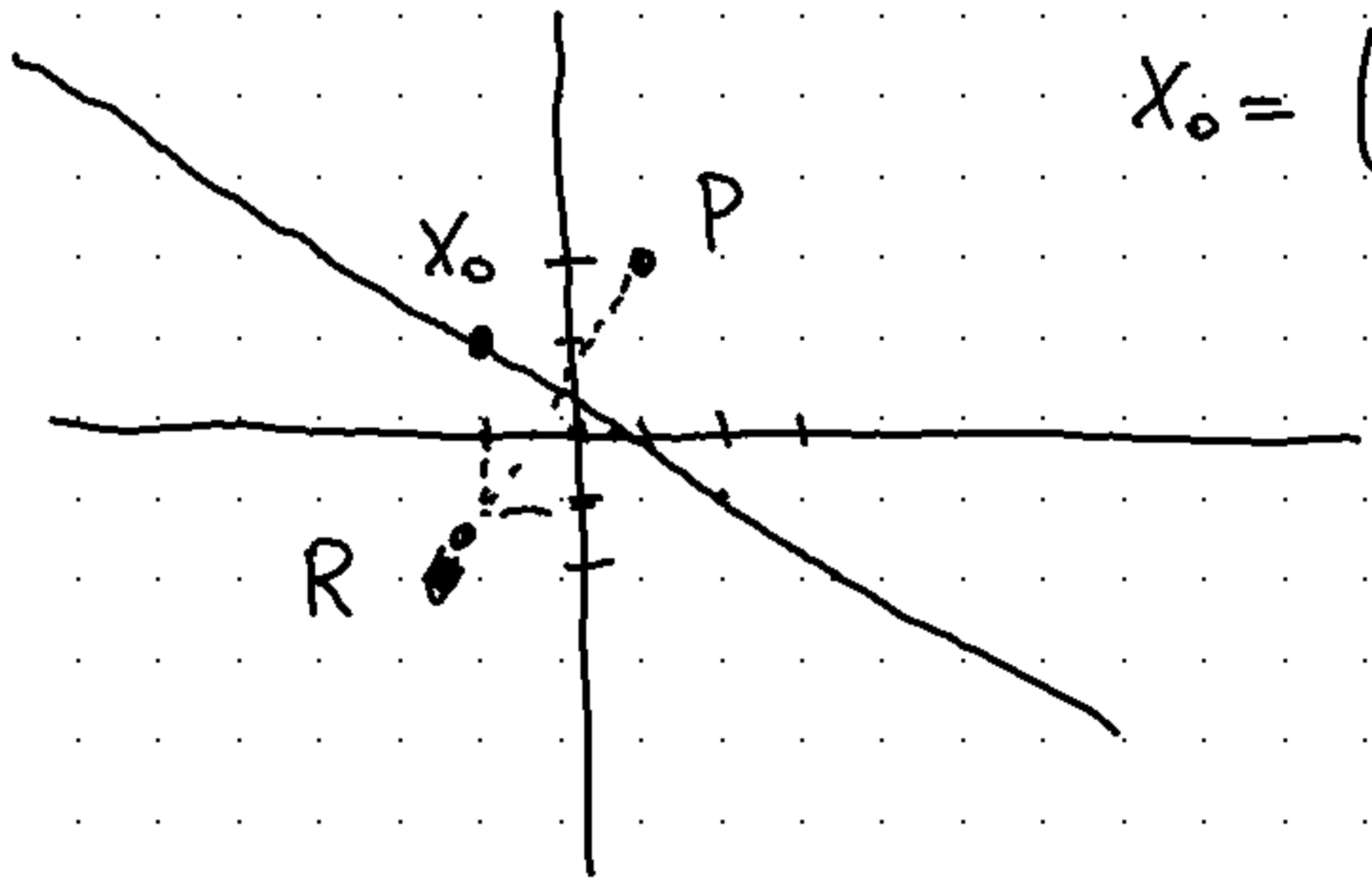


Es: $\quad r: \; 2x + 3y = 1$

$$P = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

Calcolare la riflessione ortogonale attraverso r di P.

Sol.: $\qquad X_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$$r: \; y = -\frac{2}{3}x + \frac{1}{3} \qquad m = -\frac{2}{3}$$

$$Q_m = \frac{1}{1+m^2} \begin{pmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{pmatrix}$$

$$\underset{m=-\frac{2}{3}}{=} \frac{1}{1+\frac{4}{9}} \begin{pmatrix} 1-\frac{4}{9} & -\frac{4}{3} \\ -\frac{4}{3} & \frac{4}{9}-1 \end{pmatrix}$$

$$= \frac{9}{13} \begin{pmatrix} \frac{5}{9} & -\frac{12}{9} \\ -\frac{12}{9} & -\frac{5}{9} \end{pmatrix}$$

$$= \frac{1}{13} \begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix}$$

$$P = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad X_0 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$R = X_0 + Q_m (P - X_0)$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} 5 & -12 \\ -12 & -5 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \frac{1}{13} \begin{pmatrix} -2 \\ -29 \end{pmatrix} = \begin{pmatrix} -15/13 \\ -16/13 \end{pmatrix}$$