VISCOSITY SOLUTIONS AND OPTIMAL CONTROL PROBLEMS

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1. Introduction

A *control problem* may be described as a process to influence the behavior of a dynamical system, in order to achieve a desired result.

If the goal is to minimize a *cost function* then we speak of an *optimal control problem*. More generally, in the method of *dynamical programming* we use the notions of the *value function* and the *optimal strategy*. The value function satisfies, at least formally, a first-order partial differential equation, the so-called *Hamilton-Jacobi-Bellman* equation. These equations allow us to determine the value function. Under some hypotheses of regularity, we study how to find the optimal strategy by using the value function.

In these notes we present some known results, contained in the references, arranged by the author in order to furnish a first introduction into this theory.

2. Examples of optimal control problems

We begin by giving four examples. The first three are simple and serve for illustration, while the last one will require a general method.

A. Minimal exit time from an open set. Consider a physical system satisfying the *state equation*

$$\dot{X}(s) = \alpha(s)$$

in the open interval $\Omega = (-1, 1)$, with the *initial condition*

$$X(0) = x.$$

We only consider bounded *controls* α:

$$|\alpha(s)| \leq 1 \quad \text{for all} \quad s.$$

Such a control is called *admissible.*

Problem: find α such that the system attains the boundary of Ω in the smallest possible time $T(x)$.

Proposition 2.1.

(a) We have $T(x) = 1 - |x|$ for all $x \in [-1, 1]$.

(b) For each fixed $x \in [-1, 1]$, $x \neq 0$ an optimal control is the constant function

$$\alpha(s) = \text{sign of } x, \quad 0 \leq s \leq T(x).$$

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Proof. If $0 \le t < 1 - |x|$, then for every admissible control α we have

$$|X_x^\alpha(t)| = \left| x + \int_0^t \alpha(s)\ ds \right| \le |x| + |t| < 1,$$

whence

$$T(x) \ge 1 - |x|.$$

Moreover, for $x \ne 0$ we have equality in the above estimate if and only if $t = 1 - |x|$ and $\alpha(s) =$ sign of x for all $0 \le s \le t$. □

Remarks.

- The proof shows that for $x \ne 0$ the control is unique, and it depends on the time only via the system

  $$\alpha(s) = \text{sign of } X(s).$$

  Controls of this type, called *feedback controls*, have much interest in the applications because they allow us to modify the state of the system on the basis of the sole knowledge of its actual state.
- In case $x = 0$ there are two optimal controls: the constant functions $\alpha = 1$ and $\alpha = -1$.

B. Exact controllability in minimal time. Now consider the dynamical system

$$\dot{X}(s) = \alpha(s), \quad X(0) = x$$

in $\mathbb{R}$, where we denote again by $\alpha(s)$ the control acting on the system. For example, $X(s)$ may represent the angular velocity of a system and $\alpha(s)$ the torque applied to the system. Assume that the torque is bounded:

$$|\alpha(s)| \le 1 \quad \text{for all} \quad s.$$

Problem: find α such that the system arrives to rest in the smallest possible time $T(x)$.

Proposition 2.2.

(a) We have $T(x) = |x|$ for all $x \in \mathbb{R}$.

(b) For each fixed $x \in \mathbb{R}$, the constant function

$$\alpha(s) = -\text{sign of } x, \quad 0 \le s \le T(x)$$

is the unique optimal control.

Proof. If $0 \le t < |x|$, then for every admissible control α we have

$$|X_x^\alpha(t)| = \left| x + \int_0^t \alpha(s)\ ds \right| \ge |x| - |t| > 0,$$

so that

$$T(x) \ge |x|.$$

For $t = |x|$ we have

$$|X_x^\alpha(t)| = \left| x + \int_0^t \alpha(s)\ ds \right| \ge |x| - |t| = 0$$

with equality if and only if $\alpha(s) = -$sign of x for $0 \le s \le t$. □

Remark. The optimal control is again a feedback control:

$$\alpha(s) = -\text{sign of } X(s).$$

C. A problem with integral cost. Consider the dynamical system

$$\dot{X}(s) = -X(s) \cdot \alpha(s), \quad X(0) = x$$

with bounded controls:

$$|\alpha(s)| \le 1.$$

Consider the cost function

$$u(x) = \inf_{\alpha} \int_0^{\infty} |X_x^{\alpha}(s)| e^{-2s} \, ds,$$

where $X_x^{\alpha}(t)$ denotes the state depending on the control α and of the initial position x.

Problem: find the function $u(x)$ and the corresponding optimal control.

Proposition 2.3.
(a) We have $u(x) = |x|/3$ for all $x \in \mathbb{R}$.

(b) The unique optimal control is the costant function $\alpha = 1$.

Proof. We have for every admissible control α the inequality

$$|X_x^{\alpha}(t)| = \left| x e^{-\int_0^t \alpha(s) \, ds} \right| \ge |x| e^{-t}$$

for all $t \ge 0$; hence

$$\int_0^{\infty} |X_x^{\alpha}(t)| e^{-2t} \, dt \ge \int_0^{\infty} |x| e^{-3t} \, dt = |x|/3.$$

We have equality if and only if $\alpha(s) = 1$ for all s. $\square$

D. Another problem with integral cost. Consider again the dynamical system

$$\dot{X}(s) = -X(s) \cdot \alpha(s), \quad X(0) = x$$

with bounded controls:

$$|\alpha(s)| \le 1,$$

but now with the cost function

$$u(x) = \inf_{\alpha} \int_0^{\infty} \big(|X_x^{\alpha}(s)| + |\alpha(s)| \big) e^{-2s} \, ds.$$

Problem: find the function $u(x)$ and the corresponding optimal control.

It seems to be difficult to solve this problem directly. We will solve it later by applying the theory we develop in the following sections.

General description. In general, an optimal control problem will be given by a system of ordinary differential equations

$$\dot{X}(s) = b(X(s), \alpha(s)), \quad X(0) = x,$$

where b is a given function and α is a control chosen from some given set $\mathcal{A}$ of *admissible controls.* In an optimal control problem we want to minimize a given *cost function* J by using such controls. In other words, we want to determine the *value function*

$$u(x) = \inf_{\alpha \in \mathcal{A}} J(x, \alpha).$$

Let us rewrite in this form the preceding examples. Let us choose for $\mathcal{A}$ the set of the *piecewise constant* functions $\alpha : [0, \infty) \to [-1, 1]$.

A. Exit time:

$$\begin{aligned} b(X, \alpha) &= \alpha \quad \text{for} \quad X, \alpha \in \mathbb{R}, \\ J(x, \alpha) &= \min\{t \geq 0 \ : \ |X_x^\alpha(t)| = 1\}. \end{aligned}$$

B. Exact controllability:

$$\begin{aligned} b(X, \alpha) &= \alpha \quad \text{for} \quad X, \alpha \in \mathbb{R}, \\ J(x, \alpha) &= \min\{t \geq 0 \ : \ X_x^\alpha(t) = 0\}. \end{aligned}$$

C. First problem with integral cost:

$$\begin{aligned} b(X, \alpha) &= -X \cdot \alpha \quad \text{for} \quad X, \alpha \in \mathbb{R}, \\ J(x, \alpha) &= \int_0^\infty |X_x^\alpha(s)| e^{-2s} \, ds. \end{aligned}$$

D. Second problem with integral cost:

$$\begin{aligned} b(X, \alpha) &= -X \cdot \alpha \quad \text{for} \quad X, \alpha \in \mathbb{R}, \\ J(x, \alpha) &= \int_0^\infty \big(|X_x^\alpha(s)| + |\alpha(s)|\big) e^{-2s} \, ds. \end{aligned}$$

3. Ordinary differential equations with measurable data

In order to treat the the solutions of the above introduced dynamical systems, we give here a simplified version of a theorem of Caratheodory [3] on solutions of equations of the type

$$\dot{X}(s) = b(X(s), s), \quad X(0) = x,$$

where $b : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is not necessarily continuous. Assume that

- b is measurable with respect to $s \in [0, \infty)$ for each fixed $x \in \mathbb{R}$;
- b is Lipschitzian with respect to $x \in \mathbb{R}$, for almost every fixed $s \in [0, \infty)$, with a nonnegative constant L, independent of s:

  $$|b(x, s) - b(x', s)| \leq L|x - x'|;$$

- there exist two constants M and N such that

  $$|b(0, s)| \leq M e^{Ns}$$

  for almost all $s \in [0, \infty)$.

Theorem 3.1. *For every given $x \in \mathbb{R}$, there exists a unique continuous function $X : [0,\infty) \to \mathbb{R}$ such that*

$$X(t) = x + \int_0^t b(X(s), s)\ ds, \quad t \in [0,\infty). \tag{3.1}$$

Proof. If X solves (3.1), then

$$\begin{aligned}|X(t)| &\le |x| + \int_0^t |b(0,s)|\ ds + \int_0^t |b(X(s),s) - b(0,s)|\ ds\\ &\le |x| + M\int_0^t e^{Ns}\ ds + L\int_0^t |X(s)|\ ds.\end{aligned}$$

Hence, choosing a constant $K > \max\{L, N\}$, we obtain for every $t \ge 0$ the following estimate:

$$\begin{aligned}e^{-Kt}|X(t)| &\le |x| + M\int_0^t e^{-(K-N)s}\ ds + L\int_0^t e^{-K(t-s)}e^{-Ks}|X(s)|\ ds\\ &\le |x| + \frac{M}{K-N} + \frac{L}{K}\sup_{0\le s\le t} e^{-Ks}|X(s)|.\end{aligned}$$

Hence

$$\sup_{0\le s\le t} e^{-Ks}|X(s)| \le \frac{K}{K-L}\Big(|x| + \frac{M}{K-N}\Big)$$

for every $t \ge 0$, so that the function $e^{-Ks}X(s)$ is bounded for $s \ge 0$.

Thus the solution of (3.1), if exists, belongs to the Banach space U of continuous functions $u : [0,\infty) \to \mathbb{R}$ for which

$$\|u\| := \sup_{t\ge 0} e^{-Kt}|u(t)| < \infty.$$

One may readily verify that the formula

$$(Fu)(t) := x + \int_0^t b(u(s), s)\ ds, \quad t \ge 0$$

defines a Lipschitzian map $F : U \to U$ with the Lipschitz constant $L/K < 1$. Consequently, F admits a unique fixed point. We conclude by observing that the solutions of (3.1) are exactly the fixed points of F. □

4. Principle of dynamic programming

The principle of dynamic programming, as stated by Bellman in his book Dynamic Programming published in 1957, is the following:

An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

A car race is a process satisfying the principle of dynamic programming: in order to win, we have to run the fastest possible during the whole time.

A function which satisfies the principle of dynamic programming is the so-called *distance function*: in every intermediate point, we always realize the shortest possible path.

The basic idea of dynamic programming, due to Bellman, appeared between 1949 and 1951. His first work on this subject was published in 1952, followed by a monography in 1953. Let us apply this principle to our examples.

A. Problem of exit time. Let us "forget" the proposition 2.1, and let us prove a somewhat weaker result, but by using a general method.

Proposition 4.1. *The function $T : [-1, 1] \to \mathbb{R}$ satisfies the following conditions:*

(a) $T(-1) = T(1) = 0$*;*

(b) T *is Lipschitzian;*

(c) $|T'(x)| - 1 = 0$ *in every point* $x \in (-1, 1)$ *where* T *is differentiable and* $T(x) > 0$.

Proof.
(a) Obvious from the definition.

For the proof of (b) and (c), observe that the principle of dynamic programming yields

$$T(x) = \inf_{\alpha}[T(X_x^\alpha(t)) + t] \quad \text{for every} \quad 0 \le t \le T(x). \tag{4.1}$$

(b) We prove that

$$|T(x) - T(y)| \le |x - y| \tag{4.2}$$

for every $x, y \in [-1, 1]$. Assume by symmetry that $T(x) \ge T(y)$.
The case $T(x) \le |x - y|$ is obvious:

$$T(x) - T(y) \le T(x) \le |x - y|.$$

If $T(x) > |x - y| =: t$, then take an admissible control α such that

$$\alpha(s) = \text{sign of } (y - x) \quad \text{for} \quad 0 \le s \le t := |x - y|.$$

Then $X_x^\alpha(t) = y$, so that, applying (4.1) we obtain $T(x) \le t + T(y)$, i.e., (4.2).

(c) For every sufficiently regular admissible control we have

$$X_x^\alpha(t) = x + \int_0^t \alpha(s)\ ds = x + at + o(t) = x + o(1), \quad a := \alpha(0^+),$$

and hence

$$T(X_x^\alpha(t)) = T(x) + T'(x)at + o(t)$$

if $t \searrow 0$. Choosing a constant control $\alpha = a$ and using these relations, we deduce from (4.1) the estimate

$$T(x) \le T(X_x^\alpha(t)) + t = T(x) + T'(x)at + o(t) + t,$$

whence

$$-aT'(x) - 1 \le o(1).$$

Letting $t \searrow 0$ and then maximizing with respect to a, we conclude that

$$|T'(x)| - 1 \le 0.$$

In order to show the inverse inequality, fix $0 < t < T(x)$ and $\varepsilon > 0$ arbitrarily. Using (4.1), there exists an admissible control α such that

$$T(x) > t + T(X_x^\alpha(t)) - \varepsilon t.$$

Assuming that this control is regular, using the above estimate of $T(X_x^\alpha(t))$ it follows that

$$T(x) > t + T(x) + T'(x)at + o(t) - \varepsilon t,$$

whence

$$-aT'(x) - 1 > o(1) - \varepsilon.$$

Maximizing with respect to a, this yields the inequality

$$|T'(x)| - 1 > o(1) - \varepsilon.$$

Finally, letting $t \to 0$ and then $\varepsilon \to 0$, we conclude that

$$|T'(x)| - 1 \ge 0. \quad \square$$

Remark. In the proof of the second inequality we assumed that the controls are regular. This can be avoided by an indirect argument, contained in several references cited at the end of these notes. However, we prefered to give a direct and more transparent proof.

The same remark also applies to the proof of proposition 4.4 below.

B. Problem of exact controllability. "Forgetting" the proposition 2.2 and applying the previous method we obtain the following result (the proof is left as an exercise):

Proposition 4.2. *The function $T : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(a) $T(0) = 0$*;*

(b) T *is Lipschitzian;*

(c) $1 - |T'(x)| = 0$ *in every point* $x \neq 0$ *where* T *is differentiable.*

C. First problem with integral cost. Let us "forget" the proposition 2.3. For the present problem the principle of dynamic programming may be formulated as

$$u(x) = \inf_{\alpha} \Big(\int_0^t |X_x^\alpha(s)| e^{-2s}\, ds + u(X_x^\alpha(t)) e^{-2t} \Big)$$

for every $t > 0$. We deduce from these relations the

Proposition 4.3. *The value function $u : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(a) u *is Lipschitzian;*

(b) in every point $x \neq 0$ *where* u *is differentiable, we have*

$$2u(x) + |x| \cdot (|u'(x)| - 1) = 0.$$

The proof is left as an exercise; in case of difficulty we suggest to read the next example.

D. Second problem with integral cost. In this case the principle of dynamic programming means that

$$u(x) = \inf_{\alpha} \Big(\int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big) e^{-2s}\, ds + u(X_x^\alpha(t)) e^{-2t} \Big) \tag{4.3}$$

for every $t > 0$. Hence we deduce the

Proposition 4.4. *The value function $u : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions:*

(a) u *is Lipschitzian;*

(b) in every point $x \neq 0$ *where* u *is differentiable, we have*

$$2u(x) - |x| + \max_{|a| \le 1} \{ a x u'(x) - |a| \} = 0.$$

Proof.

(a) For $x, y \in \mathbb{R}$ and $\varepsilon > 0$ fixed arbitrarily, there exists an admissible control such that

$$u(x) > \int_0^\infty \big(|X_x^\alpha(s)| + |\alpha(s)|\big) e^{-2s}\ ds - \varepsilon.$$

Since

$$u(y) \le \int_0^\infty \big(|X_y^\alpha(s)| + |\alpha(s)|\big) e^{-2s}\ ds,$$

we have

$$\begin{aligned}
u(y) - u(x) &< \int_0^\infty \big(|X_y^\alpha(s)| - |X_x^\alpha(s)|\big) e^{-2s}\ ds + \varepsilon \\
&\le \int_0^\infty |X_y^\alpha(s) - X_x^\alpha(s)| e^{-2s}\ ds + \varepsilon \\
&= |y - x| \int_0^\infty e^{-\int_0^s \alpha(t)\ dt} e^{-2s}\ ds + \varepsilon \\
&\le |y - x| \int_0^\infty e^{-s}\ ds + \varepsilon \\
&= |y - x| + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \to 0$ and using the symmetry between x and y, we conclude that

$$|u(y) - u(x)| \le |y - x| \quad \text{for all} \quad x, y \in \mathbb{R}.$$

(b) For every sufficiently regular admissible control α we have

$$X_x^\alpha(t) = x e^{-\int_0^t \alpha(s)\ ds} = x - axt + o(t) = x + o(1), \quad a := \alpha(0^+),$$

and hence

$$u(X_x^\alpha(t)) = u(x) - axu'(x)t + o(t)$$

if $t \searrow 0$. Furthermore, recall that

$$e^{-2t} = 1 - 2t + o(t)$$

if $t \to 0$.

Using these relations, we deduce from (4.3) for every admissible *constant* control $\alpha = a$ that

$$\begin{aligned}
u(x) &\le \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big) e^{-2s}\ ds + u(X_x^\alpha(t)) e^{-2t} \\
&= \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big) e^{-2s}\ ds + u(x) - axu'(x)t - 2u(x)t + o(t),
\end{aligned}$$

whence

$$2u(x) + axu'(x) \le \frac{1}{t} \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big) e^{-2s}\ ds + o(1).$$

Letting $t \to 0$ we obtain that

$$2u(x) - |x| + \{axu'(x) - |a|\} \le 0.$$

Maximizing with respect to a, we conclude that

$$2u(x) - |x| + \max_{|a| \le 1} \{axu'(x) - |a|\} \le 0.$$

In order to show the inverse inequality, fix $t > 0$ and $\varepsilon > 0$ arbitrarily. Using (4.3) there exists an admissible control α such that

$$u(x) > \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(X_x^\alpha(t))e^{-2t} - \varepsilon t.$$

Assuming for simplicity that this control is sufficiently regular, using the above estimates of $X_x^\alpha(t)$, $u(X_x^\alpha(t))$ and e^{-2t}, it follows that

$$u(x) > \int_0^t |x| + |a| + o(1)\ ds + u(x) - axu'(x)t - 2u(x)t + o(t) - \varepsilon t,$$

so that

$$2u(x) - |x| + \{axu'(x) - |a|\} \geq o(1) - \varepsilon.$$

Maximizing with respect to a this yields

$$2u(x) - |x| + \max_{|a|\leq 1}\{axu'(x) - |a|\} \geq o(1) - \varepsilon.$$

Now letting $t \to 0$ and then letting $\varepsilon \to 0$ we conclude that

$$2u(x) - |x| + \max_{|a|\leq 1}\{axu'(x) - |a|\} \geq 0. \quad \square$$

Remark. In 1951, Isaacs introduced a *tenet of transition*, a concept related to Bellman's optimality principle. Applying this principle to cases where the state equations are ordinary differential equtions, in 1954 Bellman derived the nonlinear partial differential equations which today are called Hamilton–Jacobi–Bellman equations.

For the problem of exit time from an open set, and for the two infinite horizon optimal control problem considered above, the Hamilton-Jacobi-Bellman equations are as follows:

$$\begin{aligned}
&|T'(x)| - 1 = 0 \quad \text{in} \quad x \in (-1, 1);\\
&2u(x) + |x| \cdot (|u'(x)| - 1) = 0 \quad \text{in} \quad x \in \mathbb{R};\\
&2u(x) - |x| + \max_{|a|\leq 1}\{axu'(x) - |a|\} = 0 \quad \text{in} \quad x \in \mathbb{R}.
\end{aligned}$$

In reality, it seems that Bellman did not realize that these questions are closely related to the Hamilton–Jacobi equations in mechanics, and the names of Hamilton and Jacobi were not mentioned in this context until 1960. In 1960, Kalman determined this relation and spoke about the Hamilton–Jacobi equations of the control problem.

4.1. **Optimal controls.** As an illustration, consider the Hamilton-Jacobi-Bellman equations in the case of the deterministic control problem with infinite horizon:

$$\lambda u(x) + \max_{a\in A}\{-Du(x)b(x,a) - f(x,a)\} = 0$$

with $\lambda > 0$, and let us show how they can be used in order to determine the optimal control under the hypothesis of regularity. Assume that we have determined a continuous feedback law which realizes the maximum of the corresponding Hamilton-Jacobi-Bellman equations:

$$\lambda u(x) + \{-Du(x)b(x,a(x)) - f(x,a(x))\} = 0.$$

Computing the solutions of the ordinary differential equations yields for this feedback law the equality

$$\lambda u(X_x^\alpha(s)) + \{-Du(X_x^\alpha(s))b(X_x^\alpha, \alpha(X_x(s)) - f(X_x^\alpha, \alpha(X_x(s)))\} = 0.$$

Now we compute

$$\begin{aligned} \frac{du(X_x^\alpha(s))}{ds} &= Du(X_x^\alpha(s))b(X_x^\alpha, \alpha(X_x(s))) \\ &= -f(X_x^\alpha, \alpha(X_x(s))) + \lambda u(X_x^\alpha(s)). \end{aligned}$$

Multiplying by $e^{-\lambda t}$ and integrating between 0 and $+\infty$, it is not difficult to show that

$$u(x) = J(x, \alpha(x)),$$

so that α is optimal.

As an exercise, one can apply this result to the problems C and D with integral cost:

C. An optimal feedback control α is given by

$$\alpha(x) = |xu'(x)|$$

in every point where u is differentiable.

D. An optimal feedback control α is given by[1]

$$\alpha(x) = \text{Argmax}_{\,|a|\le 1}\{axu'(x) - |a|\}$$

in every point where u is differentiable.

For more general cases, we refer to the references where ϵ-optimal controls are used.

5. Subdifferentials and superdifferentials

It is natural to ask whether the properties established in propositions 4.1–4.4 determine the corresponding value functions in a unique way. Indeed, then we may hope to have a general method in order to find the value function in more complex problems, by solving the corresponding Hamilton–Jacobi–Bellman equations. But there are some difficulties:

Examples. Consider the problem of minimal exit time. We recall from proposition 2.1 that the function $T(x) = 1 - |x|$ in $[-1, 1]$ satisfies the properties listed in proposition 4.1. However,

- the function $u(x) := -T(x)$ also has all these properties;
- fix a finite set $A \subset [-1, 1]$ containing the points ± 1 and denote by $u_A(x)$ the distance of $x \in [-1, 1]$ from A, i.e., $u_A(x) := \text{dist}\,(x, A)$. (Observe that $T = u_A$ with $A = \{-1, 1\}$.) Then the functions u_A and $-u_A$ satisfy the properties listed in proposition 4.1.

Fortunately, the *proofs* of the propositions of the preceding section suggest a solution to this problem, because they allow us to obtain additional information in the points $x \in (-1, 1)$ where T is not differentiable. We will discuss this in the next section. But first we have to generalize the notions of the derivative of functions.

Definition. Let $u : \Omega \to \mathbb{R}$ be a function and $x \in \Omega$.

[1]This notation means that $a := \alpha$ realizes a maximum of $axu'(x) - |a|$.

- The *superdifferential* of u in x is the *set*

$$D^+u(x) := \{p \in \mathbb{R} \ : \ u(x+h) \le u(x) + ph + o(h), \quad h \to 0\}.$$

- Il *subdifferential* of u in x is the *set*

$$D^-u(x) := \{p \in \mathbb{R} \ : \ u(x+h) \ge u(x) + ph + o(h), \quad h \to 0\}.$$

For example, p belongs to the superdifferential if the graph of u is "essentially" under the straight line of equation $\ell(x+h) = u(x) + ph$ in some neighborhood of x.

Remarks.

- If u is differentiable in x, then

$$D^+u(x) = D^-u(x) = \{u'(x)\}.$$

- Conversely, if $D^+u(x) \ne \emptyset$ and $D^-u(x) \ne \emptyset$ in some *interior* point of Ω, then u is differentiable in x.
- For example, the function $u(x) = 1 - |x|$ satisfies

$$D^+u(0) = [-1,1] \quad \text{and} \quad D^-u(0) = \emptyset.$$

- It is possible that both sets are empty in certain points: consider in 0 the function

$$u(x) := \begin{cases} x\sin(1/x) & \text{if } x \ne 0, \\ 0 & \text{if } x = 0. \end{cases}$$

- (Sum of sub- and superdifferentials.) If $p \in D^+u(x)$ and $q \in D^+v(x)$, then $p + q \in D^+(u+v)(x)$, i.e.,

$$D^+u(x) + D^+v(x) \subset D^+(u+v)(x).$$

  Analogously,

$$D^-u(x) + D^-v(x) \subset D^-(u+v)(x).$$

The notion of sub- and superdifferentials allows us to generalize some basic results about differentiable functions. Let us give two examples:

Proposition 5.1. *Let $u : \Omega \to \mathbb{R}$ be a function and $x \in \Omega$.*
(a) If u has a local maximum in x, then $0 \in D^+u(x)$.
(b) If u has a local minimum in x, then $0 \in D^-u(x)$.

Proof. If u has a local maximum in x, then $u(x+h) - u(x) \le 0$ for every h, close to zero. Hence

$$u(x+h) \le u(x) + 0 \cdot h + o(h)$$

for $h \to 0$ and thus $0 \in D^+u(x)$.

The other case is similar. □

Proposition 5.2. *Let $u \in C[a,b]$. There exist $x \in (a,b)$ and $p \in D^+u(x) \cup D^-u(x)$ such that*

$$u(b) - u(a) = p(b-a).$$

Proof. Changing $u(x)$ to $u(x)-cx$ with $(u(b)-u(a))/(b-a)$ and using the addition properties of sub- and superdifferentials, we may assume that $u(a)=u(b)$. We have to find $x\in(a,b)$ such that $0\in D^+u(x)\cup D^-u(x)$.

If u is constant, then every $x\in(a,b)$ has this property because $u'(x)=0$. Otherwise, applying a classical theorem of Weierstrass, u has a global minimum or a global maximum in some $x\in(a,b)$. Then we conclude by applying the preceding proposition. $\square$

Question: Which is the point x in the case of the function $u(x)=1-|x|$ in $[-1,1]$?

In order to determine more easily the sets $D^+u(x)$ e $D^-u(x)$, les us introduce the *Dini derivatives*:

$$\Lambda_-u(x)=\limsup_{h\to 0^-}\frac{u(x+h)-u(x)}{h},\quad \Lambda_+u(x)=\limsup_{h\to 0^+}\frac{u(x+h)-u(x)}{h},$$
$$\lambda_-u(x)=\liminf_{h\to 0^-}\frac{u(x+h)-u(x)}{h},\quad \lambda_+u(x)=\liminf_{h\to 0^+}\frac{u(x+h)-u(x)}{h}.$$

We always have

$$\lambda_+u(x)\le\Lambda_+u(x)\quad\text{and}\quad\lambda_-u(x)\le\Lambda_-u(x),$$

and all four Dini derivatives are equal to $u'(x)$ if u is differentiable in x.

Proposition 5.3. *The following equalities hold true:*

$$D^+u(x)=\{p\in\mathbb{R}\ :\ \Lambda_+u(x)\le p\le\lambda_-u(x)\}$$

and

$$D^-u(x)=\{p\in\mathbb{R}\ :\ \Lambda_-u(x)\le p\le\lambda_+u(x)\}.$$

Proof. Since the two relations are analogous, we only prove the first one. Dividing by h and considering separately the cases $h>0$ e $h<0$, the relation

$$u(x+h)\le u(x)+ph+o(h),\quad h\to 0$$

is equivalent to the two relations

$$\frac{u(x+h)-u(x)}{h}\le p+\frac{o(h)}{h},\quad h\to 0^+$$

and

$$\frac{u(x+h)-u(x)}{h}\ge p+\frac{o(h)}{h},\quad h\to 0^-.$$

We conclude by observing that these last two relations are equivalent to $\Lambda_+u(x)\le p$ and $p\le\lambda_-u(x)$, respectively. $\square$

6. Viscosity solutions

Let us return to the problem of exit time. We have the following generalization of proposition 4.1:

Proposition 6.1. *The function $T:[-1,1]\to\mathbb{R}$ has the following properties:*

(a) $T(-1)=T(1)=0$;

(b) T is Lipschitzian;

(c) $|p|-1\le 0$ if $x\in(-1,1)$ and $p\in D^+T(x)$;

(d) $|p|-1\ge 0$ if $x\in(-1,1)$ and $p\in D^-T(x)$.

Proof. We already know (a) and (b). For (c) e (d), it suffices to observe that in every point $x \neq 0$ we have

$$D^+T(x) = D^-T(x) = T'(x) = \pm 1,$$

while in $x = 0$ we have already seen that

$$D^+T(0) = [-1, 1] \quad \text{and} \quad D^-T(0) = \emptyset;$$

,

hence (c) and (d) follow. □

Example. Among all functions $u(x) := \pm T(x)$ and $\pm u_A(x)$, considered at the beginning of the preceding section, only $T(x) = 1 - |x|$ satisfies property (d) of proposition 6.1.

The proposition 4.1 suggests a notion of weak solution. Consider a more general case. By *Hamilton–Jacobi–Bellman equations* we understand a class of first-order nonlinear partial differential equations of the type

$$F(x, u(x), Du(x)) = 0, \tag{6.1}$$

where Ω is an open set of $\mathbb{R}^n$ and $F : \Omega \times \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$ is a continuous function. In what follows, Du denotes the *gradient vector*

$$Du(x) = (u_{x_1}, \cdots u_{x_n})$$

of a function u defined in Ω.

Example.

$$|Du(x)| - 1 = 0, \quad x \in \mathbb{R}^n.$$

Let us begin by generalizing the sub- and superdifferentials.

Definition. Let $u : \Omega \to \mathbb{R}$ be a function and $x \in \Omega$.

- The *subdifferential* of u in x is the set
$$D^+u(x) := \{p \in \mathbb{R}^n \ : \ u(x+h) \leq u(x) + p \cdot h + o(h), \quad h \to 0\}.$$
- The *superdifferential* of u in x is the set
$$D^-u(x) := \{p \in \mathbb{R}^n \ : \ u(x+h) \geq u(x) + p \cdot h + o(h), \quad h \to 0\}.$$

Remarks.

- As in the one-dimensional case, if u is differentiable in x, then
$$D^+u(x) = D^-u(x) = \{Du(x)\}.$$
- Conversely, if $D^+u(x) \neq \emptyset$ and $D^-u(x) \neq \emptyset$, then u is differentiable in x.
- Proposition 5.1 and its proof remain valid in dimension n.

Now we introduce, following Crandall and Lions [4], the

Definition. $u \in C(\Omega)$ is a *viscosity solution* of (6.1) if

$$F(x, u(x), p) \leq 0 \quad \text{for every} \quad x \in \Omega \quad \text{and} \quad p \in D^+u(x), \tag{6.2}$$

and

$$F(x, u(x), p) \geq 0 \quad \text{for every} \quad x \in \Omega \quad \text{and} \quad p \in D^-u(x). \tag{6.3}$$

Remarks.

- If u is differentiable in a point x, then (6.2) and (6.3) are equivalent to $F(x, u(x), Du(x)) = 0$.
- More generally, $u \in C(\Omega)$ is called a *(viscosity) subsolution* of (6.1) if (6.2) is satisfied, and a *(viscosity) supersolution* of (6.1) if (6.3) is satisfied.
- Later (in proposition 8.3) we give a useful equivalent definition by using test functions.

Proposition 6.2.

A. Exit time. *The minimal exit time is a Lipschitzian viscosity solution of the equation*

$$|u'(x)| - 1 = 0 \quad in \quad (-1,1), \qquad u(-1) = u(1) = 0.$$

B. Exact controllability. *The minimal time of exact controlability is a Lipschitzian viscosity solution of the equation*

$$1 - |u'(x)| = 0 \quad in \quad \mathbb{R}, \qquad u(0) = 0.$$

C. First problem with integral cost. *The value function u is a viscosity solution of the equation*

$$2u(x) + |x| \cdot \big(|u'(x)| - 1\big) = 0 \quad in \quad \mathbb{R}.$$

D. Second problem with integral cost. *The value function u is a viscosity solution of the equation*

$$2u(x) - |x| + \max_{|a| \le 1}\{axu'(x) - |a|\} = 0 \quad in \quad \mathbb{R}. \tag{6.4}$$

Proof. The first result follows from proposition 6.1; **B** and **C** can be proved similarly. Since the value function is still unknown for the problem **D**, we prove the corresponding result by another, general method, based on the principle of dynamic programming

$$u(x) = \inf_{\alpha}\Big(\int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(X_x^\alpha(t))e^{-2t}\Big), \quad t > 0. \tag{6.5}$$

We can adapt the proof of part (b) of proposition 4.4 as follows.

We recall that we have for every sufficiently regular admissible control α the relation

$$X_x^\alpha(t) = xe^{-\int_0^t \alpha(s)\ ds} = x - axt + o(t) = x + o(1), \quad a := \alpha(0^+).$$

Hence

$$u(X_x^\alpha(t)) \le u(x) - axpt + o(t)$$

if $t \searrow 0$, whenever $p \in D^+u(x)$. Since

$$e^{-2t} = 1 - 2t + o(t)$$

if $t \to 0$, using these relations, we deduce from (6.5) for every admissible *constant* control $\alpha = a$ that

$$\begin{aligned} u(x) &\le \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(X_x^\alpha(t))e^{-2t} \\ &\le \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(x) - axpt - 2u(x)t + o(t), \end{aligned}$$

whence

$$2u(x) + axp \le \frac{1}{t}\int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + o(1).$$

Letting $t \to 0$ we obtain that

$$2u(x) - |x| + \{axp - |a|\} \le 0.$$

Maximizing with respect to a, we conclude that

$$2u(x) - |x| + \max_{|a|\le 1}\{axp - |a|\} \le 0.$$

This shows that u is a viscosity subsolution.

In order to show that u is also a viscosity supersolution, fix $t > 0$ and $\varepsilon > 0$ arbitrarily. Using (6.5) there exists an admissible control α such that

$$u(x) > \int_0^t \big(|X_x^\alpha(s)| + |\alpha(s)|\big)e^{-2s}\ ds + u(X_x^\alpha(t))e^{-2t} - \varepsilon t.$$

Assuming for simplicity that this control is sufficiently regular, using the above estimates of $X_x^\alpha(t)$, $u(X_x^\alpha(t))$ and e^{-2t}, it follows for every $p \in D^-u(x)$ that

$$u(x) > \int_0^t |x| + |a| + o(1)\ ds + u(x) - axpt - 2u(x)t + o(t) - \varepsilon t,$$

so that

$$2u(x) - |x| + \{axp - |a|\} \ge o(1) - \varepsilon.$$

Maximizing with respect to a this yields

$$2u(x) - |x| + \max_{|a|\le 1}\{axp - |a|\} \ge o(1) - \varepsilon.$$

Now letting $t \to 0$ and then letting $\varepsilon \to 0$ we conclude that

$$2u(x) - |x| + \max_{|a|\le 1}\{axp - |a|\} \ge 0. \quad \square$$

We end this section by illustrating on the example of (6.4), how to find a solution of a Hamilton–Jacobi–Bellman equation.

Proposition 6.3. *The formula*

$$u(x) := \begin{cases} \frac{|x|}{2} & \text{if } |x| \le 2, \\ \frac{|x|}{3} + \frac{1}{2} - \frac{2}{3x^2} & \text{if } |x| \ge 2 \end{cases} \tag{6.6}$$

defines a viscosity solution of (6.4).

Furthermore, for each x, the maximum in (6.4) *is attained for*

$$a = \begin{cases} -1 & \text{if} \quad x < -2, \\ 0 & \text{if} \quad -2 < x < 2, \\ 1 & \text{if} \quad x > 2. \end{cases} \tag{6.7}$$

Proof. Assuming that u is differentiable and $|xu'(x)| < 1$ in an open interval I, the equation (6.4) yields $a = 0$ and $u(x) = |x|/2$. This function is differentiable indeed and satisfies the condition $|xu'(x)| < 1$ in the open intervals $(-2, 0)$ and $(0, 2)$.

Now assuming that u is differentiable and $xu'(x) > 1$ in an open interval I, the equation (6.4) yields $a = 1$ and

$$2u(x) - x + xu'(x) - 1 = 0 \quad \text{in} \quad I.$$

Hence
$$\frac{d}{dx}\big(x^2u(x)\big) = x^2 + x$$
and therefore
$$u(x) = \frac{x}{3} + \frac{1}{2} + \frac{c}{x^2}$$
with some costant c. This function is differentiable indeed and satisfies the condition $xu'(x) > 1$ in the open interval $(2, \infty)$ if $c \le -2/3$. Moreover, by choosing $c = -2/3$, the continuity condition $u(2) = 1$ is also satisfied.

Analogously, we obtain that the function
$$u(x) = -\frac{x}{3} + \frac{1}{2} - \frac{2}{3x^2}$$
is differentiable and satisfies the equation (6.4) in the open interval $(-\infty, -2)$ with $a = -1$, and the continuity condition $u(-2) = 1$.

We have thus found the function (6.6). Let us verify that it is a viscosity solution of (6.4).

The function u is continuous. One can verify directly that u is differentiable and that it satisfies (6.4) in every point $x \ne 0$ in the classical sense of the derivative. and hence the conditions of sub- and supersolutions are automatically satisfied.

Finally, for $x = 0$ we have $u(x) = 0$, so that the left-hand side of (6.4) vanishes independently of the value of $p = u'(x)$. Hence conditions (6.2) and (6.3) are satisfied. □

We will prove in the following section that the function (6.6) is in fact the value function for problem **D**.

7. Uniqueness of viscosity solutions

The results of this section justify the notion of viscosity solution. They also explain the terminology of sub- and supersolutions. We begin with the last problem, left open until now.

D. Second problem with integral cost.

We recall from propositions 6.2 and 6.3 that both the value function of this problem and the function defined by the formula

$$u(x) := \begin{cases} \frac{|x|}{2} & \text{if } |x| \le 2, \\ \frac{|x|}{3} + \frac{1}{2} - \frac{2}{3x^2} & \text{if } |x| \ge 2. \end{cases} \tag{7.1}$$

are viscosity solution of the equation

$$2u(x) - |x| + \max_{|a|\le 1}\{axu'(x) - |a|\} = 0 \quad \text{in} \quad \mathbb{R}. \tag{7.2}$$

By establishing a crucial uniqueness property, we may now completely settle the problem:

Proposition 7.1.

(a) Let u be a Lipschitzian subsolution and v a Lipschitzian supersolution of the problem (7.2). *Then $u \le v$ in $\mathbb{R}$.*

(b) Consequently, the value function of this problem is given by the formula (7.1).

(c) The unique optimal control is given by the feedback law (6.7).

In order to simplify the notation, introduce the continuous function

$$H(x,p) := -\frac{|x|}{2} + \frac{1}{2}\max_{|a|\le 1}\{axp - |a|\},$$

called *Hamiltonian.* Then the problem (7.2) can be rewritten in the more compact form

$$u(x) + H(x, u'(x)) = 0 \quad \text{in} \quad \mathbb{R}. \tag{7.3}$$

Remark. Let us explain the idea of the proof. Assume that the continuous function $u - v$ admits a global minimum in some point b and a global maximum in some point c. If u and v are also differentiable in these two points, then

$$(u-v)'(b) = (u-v)'(c) = 0,$$

so that

$$u'(b) = v'(b) \quad \text{and} \quad u'(c) = v'(c).$$

Therefore we deduce from the equation (7.3) that

$$u(b) = v(b) \quad \text{and} \quad u(c) = v(c),$$

i.e.,

$$(u-v)(b) = (u-v)(c) = 0.$$

Since

$$(u-v)(b) \le (u-v)(x) \le (u-v)(c)$$

for every x, by definition of b and c we conclude that $u = v$.

There are two technical difficulties here:

- it is not sure that $u - v$ has maximal and minimal values because $\mathbb{R}$ is not compact;
- even if there exist such points, it is not sure that u and v are differentiable in b and c.

We overcome these difficulties by using a penalization method.

Proof. One part (a) is established, parts (b) readily follow from propositions 6.2 and 6.3. Fix $\delta > 0$ arbitrarily. We prove the inequality $u \le v$ os part (a) in three steps.

(i) For every fixed $\varepsilon > 0$, consider the continuous function

$$w(x,y) := u(x) - v(y) - \frac{(x-y)^2}{2\varepsilon} - \frac{\delta}{2}(x^2 + y^2).$$

Since the functions u and v are Lipschitzian, they increase at most linearly at infinity, so that

$$w(x,y) \to -\infty \quad \text{if} \quad |x| + |y| \to \infty.$$

Consequently, w has a global maximum in some point $(x_\varepsilon, y_\varepsilon)$.

Then the function

$$x \mapsto u(x) - v(y_\varepsilon) - \frac{(x-y_\varepsilon)^2}{2\varepsilon} - \frac{\delta}{2}(x^2 + y_\varepsilon^2)$$

has a maximum in x_ε. Therefore

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta x_\varepsilon \in D^+ u(x_\varepsilon)$$

and hence

$$u(x_\varepsilon) + H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta x_\varepsilon\Big) \le 0$$

because u is a subsolution. Analogously, the function

$$y \mapsto -u(x_\varepsilon) + v(y) + \frac{(x_\varepsilon - y)^2}{2\varepsilon} + \frac{\delta}{2}(x_\varepsilon^2 + y)$$

has a minimum in y_ε. Consequently,

$$\frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon \in D^- v(y_\varepsilon)$$

and therefore

$$v(y_\varepsilon) + H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon\Big) \ge 0$$

because u is a supersolution. Combining the two inequalities we obtain that

$$u(x_\varepsilon) - v(y_\varepsilon) \le H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon\Big) - H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta y_\varepsilon\Big).$$

For every fixed x, using the relation

$$w(x, x) \le w(x_\varepsilon, y_\varepsilon)$$

we have

$$\begin{aligned} u(x) - v(x) - \delta x^2 &\le u(x_\varepsilon) - v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} - \frac{\delta}{2}(x_\varepsilon^2 + y_\varepsilon^2) \\ &\le u(x_\varepsilon) - v(y_\varepsilon) \end{aligned}$$

and hence

$$u(x) - v(x) - \delta x^2 \le H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} - \delta y_\varepsilon\Big) - H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon} + \delta x_\varepsilon\Big). \tag{7.4}$$

(ii) Next we prove that the three sequences

$$(x_\varepsilon), \quad (y_\varepsilon) \quad \text{and} \quad \Big(\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big)$$

are bounded. The relation

$$w(0, 0) \le w(x_\varepsilon, y_\varepsilon)$$

implies the inequality

$$u(0) - v(0) \le u(x_\varepsilon) - v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} - \frac{\delta}{2}(x_\varepsilon^2 + y_\varepsilon^2).$$

Consequently, denoting by L a Lipschitz constant of both u and v, we have

$$\frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} + \frac{\delta}{2}(x_\varepsilon^2 + y_\varepsilon^2) \le u(x_\varepsilon) - u(0) + v(0) - v(y_\varepsilon) \le L(|x_\varepsilon| + |y_\varepsilon|).$$

Hence

$$(|x_\varepsilon| + |y_\varepsilon|)^2 \le 2(x_\varepsilon^2 + y_\varepsilon^2) \le \frac{4L}{\delta}(|x_\varepsilon| + |y_\varepsilon|)$$

and therefore

$$|x_\varepsilon| + |y_\varepsilon| \le \frac{4L}{\delta}. \tag{7.5}$$

Now using the inequality

$$w(x_\varepsilon, x_\varepsilon) + w(y_\varepsilon, y_\varepsilon) \le 2w(x_\varepsilon, y_\varepsilon)$$

we have

$$u(x_\varepsilon) - v(x_\varepsilon) + u(y_\varepsilon) - v(y_\varepsilon) \le 2u(x_\varepsilon) - 2v(y_\varepsilon) - \frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon}.$$

Consequently,

$$\frac{(x_\varepsilon - y_\varepsilon)^2}{2\varepsilon} \le u(x_\varepsilon) - u(y_\varepsilon) + v(x_\varepsilon) - v(y_\varepsilon) \le 2L|x_\varepsilon - y_\varepsilon|$$

and therefore

$$\Big|\frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big| \le 4L.$$

(iii) Since the function H is continuous, letting $\delta \to 0$ in (7.4) and using (7.5) we obtain for every x the inequality

$$u(x) - v(x) \le H\Big(y_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big) - H\Big(x_\varepsilon, \frac{x_\varepsilon - y_\varepsilon}{\varepsilon}\Big).$$

Observe that the arguments of H are bounded with respect to ε and that $x_\varepsilon - y_\varepsilon \to 0$ if $\varepsilon \to 0$. Since H is uniformly continuous in every compact set, leting $\varepsilon \to 0$ we conclude that

$$u(x) - v(x) \le 0$$

for every x. □

Now we turn to our other examples.

C. First problem with integral cost.

Modifying the proofs of proposition 7.1 we obtain the following result which contains proposition 2.3:

Proposition 7.2.
(a) Let u be a Lipschitzian subsolution and v a Lipschitzian supersolution of the problem

$$2u(x) + |x| \cdot \big(|u'(x)| - 1\big) = 0 \quad in \quad \mathbb{R}. \tag{7.6}$$

Then $u \le v$ in $\mathbb{R}$.

(b) The value function of this control problem is the unique *Lipschitzian viscosity solution of the problem (7.6), i.e., $u(x) = |x|/3$, $x \in \mathbb{R}$.*

(c) The unique optimal control is the constant function $\alpha = 1$.

The proof is left to the reader as an exercise.

A. Exit time.

The following result contains proposition 2.1:

Proposition 7.3.
(a) Let u be a Lipschitzian subsolution and v a Lipschitzian supersolution of the problem

$$|u'(x)| - 1 = 0 \quad in \quad (-1, 1), \qquad u(-1) = u(1) = 0. \tag{7.7}$$

Then $u \le v$ in $\mathbb{R}$.

(b) The minimal exit time is the unique *Lipschitzian viscosity solution of the problem (7.7), i.e., $u(x) = 1 - |x|$, $x \in [-1, 1]$.*

(c) If $x \neq 0$, then the unique optimal control is the constant function

$$\alpha(s) = \text{sign of } x, \quad 0 \leq s \leq 1 - |x|.$$

If $x = 0$, then there are two optimal controls: the constant functions

$$\alpha(s) = \pm 1, \quad 0 \leq s \leq 1.$$

Proof. We refer to [7] for a direct proof. □

B. Exact controllability.

This problem gives an example of non uniqueness of viscosity solutions.

Proposition 7.4.
(a) It is **not** *true that if u is a Lipschitzian subsolution and v is a Lipschitzian supersolution of the problem*

$$1 - |u'(x)| = 0 \quad \text{in} \quad \mathbb{R}, \quad u(0) = 0, \tag{7.8}$$

then $u \leq v$ in $\mathbb{R}$.

(b) It is **not** *true that the minimal exact controllability time $u(x) = |x|$ is the* unique *Lipschitzian viscosity solution of the problem* (7.8).

Proof. It suffices to observe that both functions $u(x) = x$ and $u(x) = -x$ are classical, hence also viscosity solutions of (7.8). □

Remark. Let us recall a proof that $u(x) = |x|$ is a viscosity solution of (7.8). If $x \neq 0$, then $u'(x) = 1$ so that $1 - |u'(x)| = 0$. If $x = 0$, then $D^+u(0) = \emptyset$, so that the subsolution condition is trivially satisfied, while $D^-u(0) = [-1, 1]$, so that $1 - |p| \geq 0$ for all $p \in D^-u(0)$. Hence the supersolution condition is also satisfied.

On the other hand, let us show that $u(x) = -|x|$ is *not* a viscosity solution of (7.8), because it does not satisfy the subsolution condition in $x = 0$. Indeed, we have $D^+u(0) = [-1, 1]$, so that the inequality $1 - |p| \leq 0$ does not hold true for all $p \in D^+u(0)$.

As we have just seen, changing the sign of a non differentiable viscosity solution, we do not obtain in general a new viscosity solution.

8. Test functions

The method of test functions, (see Crandall, Evans and Lions [5]), allow us to visualize more efficiently the sub- and supersolutions in certain cases. We begin by characterizing the sub- and superdifferentials in this way.

Proposition 8.1. *Let $x \in \Omega$.*

(a) If $u - \phi$ has a local maximum in x for some function $\phi \in C(\Omega)$ which is differentiable in x, then $\phi'(x) \in D^+u(x)$.

(b) Conversely, if $p \in D^+u(x)$, then there exists a function $\phi \in C^\infty(\Omega)$ such that $u - \phi$ has a local maximum in x. Moreover, we may also assume that $\phi(x) = u(x)$ and $u - \phi$ has a strict *local maximum in x.*

(c) There is an analogous characterization of $D^-u(x)$, by changing the word "maximum" to "minimum".

Proof.
(a) Applying proposition 5.1 we have $0 \in D^+(u-\phi)(x)$ and hence, using the addition rule of subdifferentials we conclude that $\phi'(x) \in D^+u(x)$.

(b) See for example in [1]. For the second part it suffices to change $\phi(y)$ to

$$\psi(y) := \phi(y) + (u(x) - \phi(x)) + |y-x|^2.$$

(c) Consider the function $v(x) := -u(x)$ because $D^-u(x) = -D^+v(x)$. □

Remark. This characterization is useful from a geometrical point of view. Indeed, we may restrict ourselves to test functions which intersect the function u in x, and which remain above u (in case of maximum) or below u (in case of minimum).

We gave in section 5 an example where $D^+u(x) = D^-u(x) = \emptyset$ for a certain point. In fact, "few" such points can exist:

Proposition 8.2. *The points where the subdifferential or the superdifferential is not empty, form two dense subsets of* Ω.

Proof. We have to show that every set

$$B := \{x+h : |h| \le R\} \subset \Omega, \quad R > 0$$

contains a point y such that $D^+u(y)$ is not empty. (The case of $D^-u(y)$ is analogous.)

Fix

$$M > \max_B |u|$$

and consider the test function

$$\phi(x+h) := \frac{2M}{R^2}|h|^2.$$

The difference $u - \phi$ has a global maximum on the compact set B in some point y. This point is necessarily in the interior of the ball B because we have $|h| = R$ in every point $x+h$ of the boundary of B and therefore

$$(u-\phi)(x+h) = u(x+h) - 2M < u(x) = (u-\phi)(x).$$

Applying the preceding proposition we conclude that

$$\phi'(y) \in D^+u(y). \quad \square$$

Using the test functions we can give an equivalent definition of the sub- and supersolutions of an equation

$$F(x, u(x), Du(x)) = 0 \quad \text{in} \quad \Omega. \tag{8.1}$$

Proposition 8.3.
(a) A function $u \in C(\Omega)$ is a subsolution *of* (8.1) *if*

$$F(x, u(x), D\phi(x)) \le 0$$

for every $x \in \Omega$ and $\phi \in C^1(\Omega)$ such that $u - \phi$ has a local maximum in x.

(b) A function $u \in C(\Omega)$ is a supersolution *of* (8.1) *if*

$$F(x, u(x), D\phi(x)) \ge 0$$

for every $x \in \Omega$ and $\phi \in C^1(\Omega)$ such that $u - \phi$ has a local minimum in x.

Proof. Combine the definition of sub- and supersolutions with proposition 8.1. □

Finally, we prove the following *restriction property*:

Proposition 8.4. *Let $\Omega' \subset \Omega$ be two open sets in $\mathbb{R}^n$. If u is a viscosity solution of the equation*

$$F(x, u(x), Du(x)) = 0 \quad in \quad \Omega,$$

then its restriction v to Ω' is a viscosity solution of

$$F(x, v(x), Dv(x)) = 0 \quad in \quad \Omega'.$$

Proof. It suffices to observe that, by the local character of the definition of the sub- and supersolution, we have the inequalities

$$D^+v(x) = D^+u(x) \quad \text{and} \quad D^-v(x) = D^-u(x)$$

in every Ω'. □

Remark. The hypothesis that Ω' is open is important. For example (see [1]), the function $u(x) = x$ is a classical (hence also a viscosity) solution of

$$|u'(x)| - 1 = 0 \tag{8.2}$$

in $\mathbb{R}$, but it is not a supersolution of (8.2) in the subinterval $[0, \infty)$. Indeed, denoting by v this restriction and considering the test function $\phi = 0$, $u - \phi$ has a minimum in $x = 0$ but the inequality

$$|\phi'(0)| - 1 \geq 0$$

is not satisfied. We can also conclude without using the test functions: computing the superdifferential

$$D^-u(0) = (-\infty, 1]$$

and taking $p = 0$, we get that u does not satisfy the condition to be a viscosity supersolution.

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