

1. GEOMETRIC MEAN GENERALIZATION IN \mathbb{R}^n

Given n positive numbers x_1, x_2, \dots, x_n we define the geometric mean as follows

$$f(x_1, x_2, \dots, x_n) = \left(\prod_{i=1}^n x_i \right)^{\frac{1}{n}}$$

1.1. Gradient of f .

$$Df(x_1, x_2, \dots, x_n) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \dots, \frac{\partial f}{\partial x_n} \right)$$

where

$$\frac{\partial f}{\partial x_i}(x) = \frac{1}{n} (x_1 x_2 \dots x_{i-1} x_{i+1} \dots x_n)^{\frac{1}{n}-1} (x_1 x_2 \dots x_{i-1} x_{i+1} \dots x_n) \quad i = 1, \dots, n$$

1.2. Partial Second Derivative wrt to the variable x_i .

$$\begin{aligned} \frac{\partial^2 f}{\partial x_i^2}(x) &= \frac{1}{n} \left(\frac{1}{n} - 1 \right) (x_1 \dots x_n)^{\frac{1}{n}-2} (x_1 \dots x_{i-1} x_{i+1} \dots x_n)^2 = \\ &(x_1 \dots x_n)^{\frac{1}{n}} \left[\left(\frac{1}{n^2} - \frac{1}{n} \right) \frac{x_1^2 x_2^2 \dots x_{i-1}^2 x_{i+1}^2 \dots x_n^2}{x_1^2 x_2^2 \dots x_n^2} \right] = \\ &(x_1 x_2 \dots x_n)^{\frac{1}{n}} \left[\frac{1}{n^2} \frac{1}{x_i^2} - \frac{1}{n} \frac{1}{x_i^2} \right] \end{aligned}$$

1.3. Partial Second Derivative wrt to the variable x_i and x_j with $j \neq i$.

$$\begin{aligned} \frac{\partial^2 f}{\partial x_j \partial x_i}(x) &= \frac{1}{n} \left(\frac{1}{n} - 1 \right) (x_1 \dots x_n)^{\frac{1}{n}-2} (x_1 \dots x_{i-1} x_{i+1} \dots x_n) (x_1 \dots x_{j-1} x_{j+1} \dots x_n) + \\ &\frac{1}{n} (x_1 \dots x_n)^{\frac{1}{n}-1} (x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} x_{j+1} \dots x_n) = \\ &(x_1 \dots x_n)^{\frac{1}{n}} \left[\left(\frac{1}{n^2} - \frac{1}{n} \right) \frac{x_1^2 \dots x_i \dots x_j \dots x_n^2}{(x_1 \dots x_n)^2} + \frac{1}{n} \frac{x_1 \dots x_{i-1} x_{i+1} \dots x_{j-1} x_{j+1} \dots x_n}{x_1 \dots x_n} \right] = \\ &(x_1 \dots x_n)^{\frac{1}{n}} \left[\left(\frac{1}{n^2} - \frac{1}{n} \right) \frac{1}{x_i x_j} + \frac{1}{n} \frac{1}{x_i x_j} \right] = (x_1 \dots x_n)^{\frac{1}{n}} \frac{1}{n^2} \frac{1}{x_i x_j} \end{aligned}$$

2. CONCAVITY AND NEGATIVE (NON POSITIVE) QUADRATIC FORM

We consider the quadratic form associated to the Hessian matrix

$$(1) \quad \sum_{j,k=1}^n \frac{\partial^2 f}{\partial x_j \partial x_k} v_j v_k = \underbrace{\frac{1}{n^2} \left(\prod_{i=1}^n x_i \right)^{\frac{1}{n}}}_{\geq 0} \underbrace{\left[\sum_{j=1}^n \frac{v_j}{x_j} \sum_{k=1}^n \frac{v_k}{x_k} - n \sum_{j=1}^n \frac{v_j^2}{x_j^2} \right]}_{\geq 0}$$

Since

$$\left(\sum_{j=1}^n \frac{v_j}{x_j} \right)^2 = \sum_{j=1}^n \frac{v_j}{x_j} \sum_{k=1}^n \frac{v_k}{x_k}$$

we have only to show

$$\left(\sum_{j=1}^n \frac{v_j}{x_j} \right)^2 \leq n \sum_{j=1}^n \frac{v_j^2}{x_j^2}$$

(see inequality on linear regression).