CONVEX FUNCTIONS

1. Convex Functions

Definition 1.1. $\Omega \subset \mathbb{R}^N$ *is a* convex *set if for any x and $y \in \Omega$,*

$$\lambda x + (1-\lambda)y \in \Omega \qquad \text{for any } \lambda \in [0,1].$$

Definition 1.2. *Let C be an open convex set. $f : C \to \mathbb{R}$ is* convex *if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y \in C, \quad \lambda \in [0,1]. \tag{1.1}$$

Definition 1.3. *f is a* strictly convex *function if in* (1.1) *we have strict inequality for $x \neq y$ and $\lambda \in (0,1)$.*

Definition 1.4. *f is a* concave *function if $-f$ is convex*

$$f(\lambda x + (1-\lambda)y) \geq \lambda f(x) + (1-\lambda)f(y) \quad \forall x, y \in C, \quad \lambda \in [0,1].$$

Theorem 1.5. *Let C be an open, convex subset of $\mathbb{R}^N$ and $f : C \to \mathbb{R}$, assume $f \in C^1(C)$. Then f is convex in $C \iff$*

$$f(x) \geq f(x_0) + Df(x_0) \cdot (x - x_0) \quad \forall x, x_0 \in C.$$

$f \in C^1(C)$, f concave in $C \iff f(x) \leq f(x_0) + Df(x_0) \cdot (x - x_0)\ \forall x, x_0 \in C$

Proof. $f \in C^1(C)$ and convex in the set $C \implies$

$$f(x) \geq f(x_0) + Df(x_0) \cdot (x - x_0)\ \forall x, x_0 \in C.$$

By the assumption of convexity

$$f(\lambda x + (1-\lambda)x_0) = f(x_0 + \lambda(x - x_0)) \leq \lambda f(x) + (1-\lambda)f(x_0).$$

This means

$$f(x_0 + \lambda(x - x_0)) - f(x_0) \leq \lambda f(x) - \lambda f(x_0),$$

$\lambda > 0$

$$\frac{f(x_0 + \lambda(x - x_0)) - f(x_0)}{\lambda} \leq \frac{\lambda f(x) - \lambda f(x_0)}{\lambda}$$

Then sending $\lambda \to 0^+$ we get the result:

$$f(x_0) + Df(x_0) \cdot (x - x_0) \leq f(x).$$

Next we assume $f(x) \geq f(x_0) + Df(x_0) \cdot (x - x_0)\ \forall x, x_0 \in C$. We show that f is convex

Change x_0 with $x_0 + \lambda(x - x_0)$ in $f(x) \geq f(x_0) + Df(x_0) \cdot (x - x_0)$.

$$f(x) \geq f(x_0 + \lambda(x - x_0)) + Df(x_0 + \lambda(x - x_0)) \cdot (x - (x_0 + \lambda(x - x_0)))$$

$$f(x) \geq f(x_0 + \lambda(x - x_0)) + Df(x_0 + \lambda(x - x_0)) \cdot (x - x_0 - \lambda(x - x_0))$$

Then

$$f(x) \geq f(x_0 + \lambda(x - x_0)) + (1 - \lambda)Df(x_0 + \lambda(x - x_0)) \cdot (x - x_0)$$

$$\lambda f(x) \geq \lambda f(x_0 + \lambda(x - x_0)) + \lambda(1 - \lambda)Df(x_0 + \lambda(x - x_0)) \cdot (x - x_0) \tag{1.2}$$

We go back to

$$f(x) \geq f(x_0) + Df(x_0) \cdot (x - x_0) \ \forall x, x_0 \in C.$$

Change x with x_0 and change x_0 with $x_0 + \lambda(x - x_0)$ in the inequality above.

$$f(x_0) \geq f(x_0 + \lambda(x - x_0)) - \lambda Df(x_0 + \lambda(x - x_0)) \cdot (x - x_0)$$

This means

(1.3)

$$(1-\lambda)f(x_0) \geq (1-\lambda)f(x_0+\lambda(x-x_0)) - (1-\lambda)\lambda Df(x_0+\lambda(x-x_0)) \cdot (x-x_0)$$

Adding (1.2) and (1.3)

$$\lambda f(x) + (1 - \lambda)f(x_0) \geq f(x_0 + \lambda(x - x_0)).$$

This show the convexity of f. □

Remark 1. *We recall that $Df(x_0) = 0$ is always a necessary condition for local optimality in an unconstrained problem. The previous theorem states that for convex problems, $Df(x_0) = 0$ is not only necessary, but also sufficient for local and global optimality (minimization problem): from*

$$f(x) \geq f(x_0) + Df(x_0) \cdot (x - x_0) \ \forall x, x_0 \in C.$$

we obtain

$$f(x) \geq f(x_0)$$