THE NUMBER e

PAOLA LORETI

Consider the set of real numbers

$$A = \{x_n = \sum_{k=0}^{n} \frac{1}{k!} \ : \ n \in \mathbb{N}\}.$$

The set A is bounded. Indeed, we have clearly

$$\sum_{k=0}^{n} \frac{1}{k!} \geq 2$$

for all n. On the other hand, since

$$k! \geq 2^{k-1}$$

for all $k \in N$ by induction, we have

$$\sum_{k=0}^{n} \frac{1}{k!} \leq 1 + \sum_{k=1}^{n} \frac{1}{2^{k-1}} <= 1 + 2 = 3$$

for all n. Hence

$$2 < \sup_n A \leq 3.$$

By definition, we set

$$e = \sup_n A.$$

Now we consider

$$A' = \{x_n = \left(1 + \frac{1}{n}\right)^n \ : \ n \in \mathbb{N}\},$$

By Bernoulli inequality

$$2 < \sup_n A'$$

and, applying the inequality

$$\sqrt[n+2]{a_1 \ldots a_{n+2}} \leq \frac{a_1 + \cdots + a_{n+2}}{n+2}$$

with

$$a_1 = \cdots = a_n = 1 + \frac{1}{n} \quad \text{and} \quad a_{n+1} = a_{n+2} = \frac{1}{2}$$

since

$$\sqrt[n+2]{\left(1 + \frac{1}{n}\right)^n \cdot \frac{1}{2} \cdot \frac{1}{2}} \leq \frac{n+2}{n+2} = 1.$$

We obtain $x_n \leq 4$. Hence (x_n) is bounded from above by 4. Then, we set

$$e' = \sup_n A'.$$

Date: June 26, 2007.

In order to give another useful definition of e, consider the binomial expansion,

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k}\frac{1}{k!}.$$

But

$$\begin{aligned} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k} &= \frac{n}{n}\cdot\frac{n-1}{n}\cdots\frac{n-k+1}{n} \\ &= 1\Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\dots\Big(1-\frac{k-1}{n}\Big) \\ &\le 1. \end{aligned}$$

Hence

$$\left(1+\frac{1}{n}\right)^n \le \sum_{k=0}^{n}\frac{1}{k!}$$

for all n. Taking the supremum over n of

$$A' = \{x_n = \left(1+\frac{1}{n}\right)^n \;:\; n \in \mathbb{N}\},$$

we conclude that

$$\sup_n A' \le \sup_n A.$$

We have just shown that

$$e' \le e.$$

We are going to show that in fact $e' = e$. We take $m < n$. Then

$$\begin{aligned} \left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k}\frac{1}{k!} \\ &\ge \sum_{k=0}^{m} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k}\frac{1}{k!}. \end{aligned}$$

By the definition of e', it follws that

$$\begin{aligned} e' &\ge \sum_{k=0}^{m} \frac{n(n-1)(n-2)\dots(n-k+1)}{n^k}\frac{1}{k!} \\ &= \sum_{k=0}^{m}\Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\dots\Big(1-\frac{k-1}{n}\Big)\cdot\frac{1}{k!}. \end{aligned}$$

Taking the suprumum over n we conclude that

$$e' \ge \sum_{k=0}^{m}\frac{1}{k!}$$

for all m. Now taking the supremum over m, the reverse inequality

$$e' \ge e$$

and thus the equality

$$e' = e$$

follows.

Let us investigate more closely the sequence

$$x_n = \left(1 + \frac{1}{n}\right)^n.$$

We are going to show that this sequence is increasing, so that the Neper number is the limit of this sequence, as n tends to ∞, as a consequence of the fondamental theorem of monotone sequences.

We prove the monotonicity of the sequence x_n is two different ways.

First we consider the ratio

$$\frac{x_2}{x_1} > 1,$$

and

$$\begin{aligned}
\frac{x_{n+1}}{x_n} &= \frac{\left(1 + \frac{1}{n+1}\right)^{n+1}}{\left(1 + \frac{1}{n}\right)^n} \\
&= \left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n}{n+1}\right)^n \\
&= \left(\frac{n+2}{n+1}\right)^{n+1} \left(\frac{n}{n+1}\right)^{n+1} \left(\frac{n+1}{n}\right) \\
&= \left(\frac{n+2}{(n+1)} \frac{n}{(n+1)}\right)^{n+1} \left(\frac{n+1}{n}\right) \\
&= \left(\frac{n^2+2n}{(n^2+2n+1)}\right)^{n+1} \left(\frac{n+1}{n}\right) \\
&= \left(\frac{n^2+2n+1-1}{(n^2+2n+1)}\right)^{n+1} \left(\frac{n+1}{n}\right) \\
&= \left(\frac{n^2+2n+1}{n^2+2n+1} - \frac{1}{n^2+2n+1}\right)^{n+1} \left(\frac{n+1}{n}\right).
\end{aligned}$$

We recall the Bernoulli inequality

$$(1+h)^n > 1 + nh$$

$h \geq -1, \quad h \neq 0, \forall n \in \mathbb{N}$, such that $\quad n \geq 2$.

We have

$$h = -\frac{1}{n^2+2n+1} > -1, \quad \forall n$$

since

$$\frac{1}{n^2+2n+1} < 1, \quad \forall n.$$

Hence,

$$\frac{x_{n+1}}{x_n} = \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^{n}} > \left(1-\frac{1}{n+1}\right)\left(\frac{n+1}{n}\right) = 1.$$

This ends the first proof. Next, we give another proof based on geometrical and arithmetic media of positive real numbers.

The sequence (x_n) defined by

$$x_n = \left(1+\frac{1}{n}\right)^n, \quad n = 1, 2, \ldots .$$

is bounded and increasing. Indeed, applying the inequality

$$\sqrt[n+1]{a_1 \ldots a_{n+1}} \le \frac{a_1 + \cdots + a_{n+1}}{n+1}$$

with

$$a_1 = \cdots = a_n = 1 + \frac{1}{n} \quad \text{and} \quad a_{n+1} = 1,$$

we obtain

$$\sqrt[n+1]{\left(1+\frac{1}{n}\right)^n} \le \frac{n+2}{n+1} = 1 + \frac{1}{n+1}.$$

This is equivalent to $x_n \le x_{n+1}$. Hence (x_n) is increasing.

References

[1] E. Giusti, *Analisi Matematica I*, Boringhieri Ed, 1988.

Dipartimento di Metodi e Modelli, Matematici per le Scienze Applicate, Università di Roma "La Sapienza", Via A. Scarpa, 16, 00161 Roma, Italy

E-mail address: loreti@dmmm.uniroma1.it