Permutations, tensor products, and Cuntz algebra automorphisms

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Abstract

We study the reduced Weyl groups of the Cuntz algebras $\mathcal{O}_n$ from a combinatorial point of view. Their elements correspond bijectively to certain permutations of n^r elements, which we call stable. We mostly focus on the case $r = 2$ and general n. A notion of rank is introduced, which is subadditive in a suitable sense. Being of rank 1 corresponds to solving an equation which is reminiscent of the Yang-Baxter equation. Symmetries of stable permutations are also investigated, along with an immersion procedure that allows to obtain stable permutations of $(n+1)^2$ objects starting from stable permutations of n^2 objects. A complete description of stable transpositions and of stable 3-cycles of rank 1 is obtained, leading to closed formulas for their number. Other enumerative results are also presented which yield lower and upper bounds for the number of stable permutations.

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Contents

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1 Introduction

The Cuntz algebras $\mathcal{O}_n$ constitute a prominent class of C^*-algebras, which has attracted much attention since their appearance in the seminal paper [10]. By now, they have been studied from several different points of view (see e.g. [16, 17, 2, 22]), and Cuntz himself computed their K-theory [12], and discussed some natural generalisations with very interesting connections with the theory of dynamical systems [13, 14]. Cuntz also started the investigation of their automorphisms [11]. These, however, remained quite elusive, with the exception of notable cases (see e.g. [25]), probably due to the difficulty in providing explicit examples. Thinking of a unital C*-algebra as (the algebra of complex-valued functions on) a compact noncommutative space, the automorphism group is nothing but the homeomorphism group of this virtual space and, as such, it is rich of hidden geometric content, with the inner automorphism playing the role of gauge symmetries. It is only over the last few years that new tools were introduced, that allowed considerable advance in spelling out some aspects of the structure of the group of $*$-automorphisms of $\mathcal{O}_n$, $\mathrm{Aut}(\mathcal{O}_n)$, as well as that of its quotient $\mathrm{Out}(\mathcal{O}_n)$ modulo inner automorphisms [9, 8, 6, 7]. As a matter of fact, as abstract groups both $\mathrm{Aut}(\mathcal{O}_n)$ and $\mathrm{Out}(\mathcal{O}_n)$ are horribly complicated, e.g. $\mathrm{Out}(\mathcal{O}_2)$ contains every second countable locally compact group, so they have to be handled with due care. However, they have a discrete shadow, rich of appealing yet quite mysterious structure, that opens a door on seemingly new challenging combinatorics. Indeed, in order to determine the properties of $\mathrm{Aut}(\mathcal{O}_n)$, in [9, 8] it is developed Cuntz's original idea of exploiting the comparison with the theory of semisimple Lie groups by introducing some kind of (reduced) Weyl groups for $\mathrm{Aut}(\mathcal{O}_n)$ as normalizers of a of maximal torus (of infinite dimension). These groups exhibit a deep combinatorial flavour, as it turns out that their elements are parametrised by certain permutations of n^r objects, $r = 1, 2, 3, \ldots$, hereafter referred to as stable permutations (by a vague analogy with the expansion of a real number as a continuous fraction in the past we have sometimes called them also rational permutations). Notice that these stable permutations do not form a group by themselves as long as one considers the usual group operation on permutations, but they do if their product is (re)defined by reproducing the composition of the corresponding automorphisms. The resulting product law is quite complicated and certainly it would not have come to mind without looking at the Cuntz algebras.

An important open problem, that motivated this research, is to count the stable permutations, with the ultimate goal of giving an explicit complete characterization.

There is empirical evidence that very few permutations are stable. For instance for $n = 3, r = 2$, among the $3^2! = 362,880$ possible permutations, only 576 of them are stable, so that $576/362880 = 0,0015873$... In order to get an idea of what is going on it would be useful to compute some more of these numbers. However, the computational problem of counting stable permutations in $S([n]^r)$

although definitely intriguing is very demanding and these numbers are known, mostly by means of massive computer calculations, only for $n + r \leq 7$ [9, 8, 1]. It would be a remarkable advance to match or relate those numbers with other numbers arising from different situations. The most optimistic hope would be to arrive at some recursive or even closed formulas, or to get the expression for the generating functions.

One might also expect that N_n^r, the number of stable permutations in $S([n]^r)$, is remarkably relatively small, i.e. $\lim_{r\to\infty} N_n^r/(n^r)! = 0$ for each n. However, a proof of this fact has yet to be found, as well as a precise claim on the speed of the decay (which is likely to be extremely fast). For this and other purposes, taking into account that to date it seems still beyond capabilities to determine those exact numbers, good estimates from below or from above would also provide valuable information.

The computational difficulties alluded to arise for two reasons. On the one hand, the number $(n^r)!$ of permutations to test becomes quickly unmanageable in any practical sense. On the other hand, given a permutation u of $[n]^r$, to decide whether it is stable, one has to go through a finite but cumbersone procedure that requires constructing a related sequence of permutations of $n^{r+1}, n^{r+2}, \ldots$ elements by combining suitable embeddings and shifts of u. A further improvement was found in [9] based on rooted trees which however still requires heavy computer calculations.

In this paper we study, from a combinatorial point of view, the stable permutations of $[n]^2$. We have decided to focus mostly on the case $r = 2$ (and arbitrary n) in order to maintain the length at a reasonable size, but we believe that this approach provides several important hints also for the case $r > 2$. In addition to the fact that most of our results hold for all n, the novelty of our approach lies in a more thorough analysis and explicit understanding of the combinatorial structure of the stable permutations.

The organization of the paper is as follows. In the next section we provide some background on Cuntz algebras and their automorphisms. In Section 3 we introduce the tensor product of two permutations and derive some elementary properties of this operation. In Section 4 we define our main object of study, namely stable permutations. We define the rank of a stable permutation, and study in more detail the stable permutations of rank 1, and those whose associated automorphism is an involution. In Section 5 we introduce and study a condition, which we call compatibility, that ensures that the (ordinary) product of two stable permutations is still stable, and give an upper bound on the rank of the product in terms of the ranks of the factors (Theorem 5.2). As a consequence of these results we determine some explicit classes of stable permutations. In Section 6 we study symmetries of stable permutations. More precisely, we study the effect on stability of the symmetries of the square grid $[n]^2$, and show that the symmetric group S_n acts in various ways on the set of stable permutations. In Section 7 we provide a combinatorial construction, that we call immersion, that produces stable permutations in $S([n+1]^2)$ from stable permutations in $S([n]^2)$ (Theorem 7.8), and preserves the property of having rank 1 (Corollary 7.5). In Section 8 we characterize some special classes of stable permutations. More precisely, we show that a stable transposition is necessarily of rank 1, and classify such transpositions (Theorem 8.1). We also characterize the 3-cycles that are stable of rank 1 (Theorem 8.7), and give a sufficient condition for a k-cycle to be stable of rank 1 (Proposition 8.5). In Section 9 we show that the arithmetic structure of n affects the number of stable permutations of $[n]^2$. More precisely, we give a combinatorial construction of stable permutations that cannot be performed if n is a prime (Propositions 9.1 and 9.2). In Section 10 we discuss, in light of the results obtained in this work, the cases $n = 3$ and $n = 4$ (with $r = 2$). In Section 11 we discuss enumerative aspects of stable permutations. More precisely, we reduce the problem of enumerating the stable permutations to that of enumerating a particular subclass of them, which we call irreducible stable permutations (Proposition 11.1), and we use the results obtained in previous sections to give upper and lower bounds for the number of stable permutations (Corollary 11.12, Proposition 11.2, Corollary 11.14). Finally, in Section 12, we discuss some conjectures, open problems, and directions for further research, that arise from the present work.

2 Preliminaries

In the sequel we adopt the following notation: for $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ we set $[n] = \{1, \dots, n\}$ and, for $h \in \mathbb{N}$, $[n]^h = [n]^{\times h}$. We also let $\binom{[n]}{k} := \{S \subseteq [n] : |S| = k\}$, and $(n)_k := k!\binom{n}{k}$ for all $k \in \{0, 1, \dots\}$. If $\{i_1, \dots, i_k\} \subseteq \mathbb{Z}$ then we write $\{i_1, \dots, i_k\}_<$ to mean that $i_1 < \dots < i_k$. For $x \in \mathbb{R}$, we let $\lfloor x \rfloor$ and $\lceil x \rceil$ be the largest integer $\leq x$ and the smallest integer $\geq x$, respectively. Given a statement P we let $\chi(P) := 1$ if P is true and $\chi(P) := 0$ otherwise. S_n denotes the symmetric group on n symbols, so $S_n := S([n])$ where $S(A) := \{u : A \to A : u \text{ is a bijection}\}$. Given $\sigma \in S_n$ we write $\sigma = a_1 \cdots a_n$ to mean that $\sigma(i) = a_i$ for all $i \in [n]$ (*one-line* notation). We also write σ in *disjoint cycle form* omitting to write the 1-cyles of σ. So for example, if $\sigma = ((1, 2), (3, 1), (2, 2))((4, 5), (2, 1)) \in S([6]^2)$ then $\sigma(1, 2) = (3, 1)$, $\sigma(3, 1) = (2, 2)$, $\sigma(2, 2) = (1, 2)$, etc... Recall that, as it is easy to see, if $\sigma = \prod_{i=1}^{j}(a_{i,1}, a_{i,2}, \dots, a_{i,k_i})$ is the disjoint cycle decomposition of $\sigma \in S_n$, and $\tau \in S_n$ then $\prod_{i=1}^{j}(\tau(a_{i,1}), \tau(a_{i,2}), \dots, \tau(a_{i,k_i}))$ is the disjoint cycle decomposition of $\tau\sigma\tau^{-1}$. We follow [28] for other notation and terminology concerning enumerative combinatorics.

For all undefined notation, terminology and basics on the theory of C^*-algebras we refer the reader to [15]. Let $n \geq 2$. Consider an infinite-dimensional (separable) complex Hilbert space $\mathcal{H}$, and let $B(\mathcal{H})$ be the set of bounded linear operators on $\mathcal{H}$. Let $S_i \in B(\mathcal{H})$, $i = 1, \ldots, n$ be isometries, i.e. such that $S_i^* S_i = 1$, that satisfy

$$\sum_{i=1}^{n} S_i S_i^* = 1 \ .$$

It is very easy to see that such operators always exist. The Cuntz algebra $\mathcal{O}_n$ is the C^*-algebra generated by the operators S_i, $i = 1, \ldots, n$ as above. It is well-known that $\mathcal{O}_n$ is independent, up to isomorphism, of the isometries S_i. Moreover, $\mathcal{O}_n$ is a simple C^*-algebra, that is it has no nontrivial closed two-sided ideal, in particular its center reduces to the complex multiples of the identity. One can describe $\mathcal{O}_n$ as the closed linear span of the Wick-ordered monomials $S_\mu S_\nu^*$ where μ, ν are arbitrary multi-indices (possibly empty) and, for a multi-index $\mu = (\mu_1, \ldots, \mu_k) \in [n]^k$, we define $S_\mu = S_{\mu_1} \ldots S_{\mu_k}$ if $k > 0$ and $S_\mu := 1$ if $k = 0$. For μ as above, we say that $k = |\mu|$ is the length of μ. One has important subalgebras

$$\mathcal{O}_n \supset \mathcal{F}_n \supset \mathcal{D}_n .$$

Here $\mathcal{F}_n$ is the closed linear span of $S_\mu S_\nu^*$ with $|\mu| = |\nu|$, and is isomorphic to the C^*-algebra infinite tensor product $\bigotimes_{i=1}^{\infty} M_n$ where M_n is the algebra of $n \times n$ complex matrices. While $\mathcal{D}_n$ is the closed linear span of $S_\mu S_\mu^*$ and is a MASA in $\mathcal{O}_n$, isomorphic to the algebra of continuous functions on $[n]^{\mathbb{N}}$ equipped with the product topology, where $[n]$ has the discrete topology, which is a Cantor set.

Denote by $\mathrm{End}(\mathcal{O}_n)$ the semigroup of unital $*$-endomorphisms of $\mathcal{O}_n$ (endomorphisms, for short). Any element u in the unitary group $\mathcal{U}(\mathcal{O}_n) = \{v \in \mathcal{O}_n \mid v^* v = v v^* = 1\}$ determines an endomorphism λ_u of $\mathcal{O}_n$ such that $\lambda_u(S_i) = u S_i$, $i = 1, \ldots, n$ and the map $u \mapsto \lambda_u$ is a bijection between $\mathcal{U}(\mathcal{O}_n)$ and $\mathrm{End}(\mathcal{O}_n)$, with inverse $u_\lambda = \sum_{i=1}^{n} \lambda(S_i) S_i^*$, $\lambda \in \mathrm{End}(\mathcal{O}_n)$. In general, this bijection does not preserve the semigroup operations, indeed it holds

$$\lambda_u \circ \lambda_v = \lambda_{\lambda_u(v) u}$$

for all $u, v \in \mathcal{U}(\mathcal{O}_n)$. If φ denotes the so-called canonical endomorphism of $\mathcal{O}_n$, given by

$$\varphi(x) = \sum_{i=1}^{n} S_i x S_i^*, \quad x \in \mathcal{O}_n \ ,$$

one then has, for $\mu = (\mu_1, \ldots, \mu_k)$,

$$\lambda_u(S_\mu) = u \varphi(u) \ldots \varphi^{k-1}(u) S_\mu \ .$$

In general, given any $u \in \mathcal{U}(\mathcal{O}_n)$ the associated λ_u is automatically injective, however deciding whether λ_u is an automorphism, i.e. surjective, is a difficult problem. In addition to the inner automorphisms $\mathrm{Ad}(u) = \lambda_{u\varphi(u^*)}$, notable exceptions are provided by $u \in \mathcal{F}_n^1$ (where, for $k \in \mathbb{N}$, $\mathcal{F}_n^k = \mathrm{span}\{S_\mu S_\nu^* \mid |\mu| = |\nu| = k\} \simeq M_{n^k}$) and $u \in \mathcal{U}(\mathcal{D}_n)$, for which the associated λ_u are always automorphisms, called quasi-free (or Bogolubov) and diagonal, respectively.

The following result has been shown in [9, Theorem 3.2] (with slightly different conventions).

Theorem 2.1. *Let $u \in \mathcal{F}_n^k$, $k \in \mathbb{N}$, then $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$ with inverse $(\lambda_u)^{-1} = \lambda_v$ for some $v \in \mathcal{F}_n^h$, $h \in \mathbb{N}$, if and only if the sequence of unitaries*

$$\{\varphi^k(u^*) \cdots \varphi(u^*)u^*\varphi(u) \cdots \varphi^k(u)\}_{k=0,1,\ldots}$$

is eventually constant, in which case its limit coincides with v.

Unitaries of the form $u = \sum_{|\mu|=|\nu|=r} S_\mu S_\nu^*$ can be identified with permutation matrices of size n^r, so with elements of $S([n]^r)$. Denote by $\mathcal{P}_n^r$ the set of such unitaries. Following [9] we define the *reduced Weyl group* of $\mathcal{O}_n$ to be the quotient

$$\Big(\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) \cap \mathrm{Aut}(\mathcal{O}_n, \mathcal{D}_n)\Big)/\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n)$$

where $\mathrm{Aut}(\mathcal{O}_n, \mathcal{F}_n) = \{\alpha \in \mathrm{Aut}(\mathcal{O}_n) \mid \alpha(\mathcal{F}_n) = \mathcal{F}_n\}$ and $\mathrm{Aut}_{\mathcal{D}_n}(\mathcal{O}_n) = \{\alpha \in \mathrm{Aut}(\mathcal{O}_n) \mid \alpha(x) = x,\ x \in \mathcal{D}_n\}$. It is known that there is a bijection between the reduced Weyl group of $\mathcal{O}_n$ and

$$\Big\{u \in \bigcup_{r\in\mathbb{N}} \mathcal{P}_n^r \mid \lambda_u \in \mathrm{Aut}(\mathcal{O}_n)\Big\} .$$

3 Tensor product of permutations

In this section we introduce the tensor product of two permutations and study some of its fundamental properties, which will be used repeatedly in the rest of this work without explicit mention. This operation corresponds to the tensor product of the associated permutation matrices.

Let n and m be two integers larger than 1. Let $u \in S_n$, $v \in S_m$. Define $u \otimes v \in S_{nm}$ by

$$(u \otimes v)((\alpha - 1)m + \beta) := (u(\alpha) - 1)m + v(\beta)$$

for all $\alpha \in [n], \beta \in [m]$. For instance, if $u = (12) \in S_2$ and $v = (132) \in S_3$ then $u \otimes v = (162435) \in S_6$. The next result is then immediate.

Lemma 3.1. *Let $u, u' \in S_n$, $v, v' \in S_m$. Then*

$$(u \otimes v)(u' \otimes v') = (uu') \otimes (vv')$$

In particular, $(u \otimes v)^{-1} = u^{-1} \otimes v^{-1}$.

Proposition 3.2. *Let $\sigma \in S_{mn}$. Then there exists $\tau \in S_m$ such that $\sigma = \tau \otimes 1$ if and only if $\sigma(j) \equiv j$ (mod n) for all $j \in [mn]$ and $\sigma(j+1) = \sigma(j) + 1$ for all $j \in [mn]$, $j \not\equiv 0$ (mod n).*

Proof. Let $\tau \in S_m$ be such that $\sigma = \tau \otimes 1$. Let $j \in [mn]$, $j = (k-1)n + i$ for some $k \in [m]$, $i \in [n]$. Then

$$\sigma(j) = (\tau \otimes 1)(j) = n(\tau(k) - 1) + i$$

so $\sigma(j) \equiv j \pmod{n}$. Furthermore, if $j \not\equiv 0 \pmod{n}$ then $i < n$. Hence $j + 1 = (k-1)n + i + 1$ and $i + 1 \in [n]$ so

$$\sigma(j+1) = (\tau(k) - 1)n + i + 1 = \sigma(j) + 1 \ .$$

Conversely, let $\sigma \in S_{mn}$ be such that $\sigma(j) \equiv j \pmod{n}$ for all $j \in [mn]$ and $\sigma(j+1) = \sigma(j) + 1$ for all $j \in [mn]$ such that $j \not\equiv 0 \pmod{n}$. Let $\tau : [m] \to [m]$ be defined by

$$\sigma\big((k-1)n + 1\big) = n(\tau(k) - 1) + 1$$

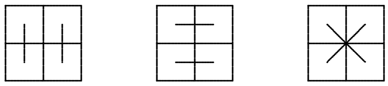
for all $k \in [m]$. Then $\tau(k) \in [m]$ and it is clear that τ is a bijection, and that $\sigma(j) = (\tau \otimes 1)(j)$ for all $j \in [mn]$ such that $j \equiv 1 \pmod{n}$. Furthermore, if $j \in [mn]$, $j = (k-1)n + i$ ($k \in [m]$, $i \in [n]$) then

$$\begin{aligned}\sigma(j) &= \sigma\big((k-1)n + \underbrace{1 + \ldots + 1}_{i}\big) = \sigma\big((k-1)n + 1\big) + i - 1 \\ &= \big(\tau(k) - 1\big)n + i = (\tau \otimes 1)(j) \ .\end{aligned}$$

□

We find it convenient to identify $S([n] \times [m])$ with S_{nm} by labeling lexicographically the elements of $[n] \times [m]$ (so $(1,1)$ is labeled 1, $(1,2)$ is labeled 2, ..., $(1,m)$ is labeled m, $(2,1)$ is labeled $m+1$, etc...) and to represent them graphically by drawing the cycles of the permutation as directed (except for the cycles of length two) cycles of the rectangular grid $[n] \times [m]$. We call this the *box diagram* (or simply *diagram*) of the permutation.

Example 3.3. The box diagrams of $3412, 2143, 4321 \in S_{2^2}$ are, respectively,



Note that $3412 = 21 \otimes 12$, $2143 = 12 \otimes 21$, $4321 = 21 \otimes 21$ (these permutations correspond to the unitaries denoted by $f, \varphi(f), f\varphi(f)$ in [9], where φ is the unilateral shift). Moreover, with these identifications, if $u \in S_n$ and $v \in S_m$ then

$$(u \otimes v)(x, y) = (u(x), v(y))$$

for all $(x, y) \in [n] \times [m]$. If $u \in S_n$, $v \in S_m$ and $w \in S_l$ one also has $((u \otimes v) \otimes w)(x, y, z) = (u \otimes (v \otimes w))(x, y, z) = (u(x), v(y), w(z))$ for all $(x, y, z) \in [n] \times [m] \times [l]$, i.e. the tensor product is associative.

Let $u \in S([n] \times [m])$. Define $u_1 : [n] \times [m] \to [n]$, and $u_2 : [n] \times [m] \to [m]$ by letting

$$u((\alpha - 1)m + \beta) =: (u_1(\alpha, \beta) - 1)m + u_2(\alpha, \beta)$$

for all $\alpha \in [n]$ and $\beta \in [m]$. If we identify $S([n] \times [m])$ with S_{nm} as explained above then

$$u(x, y) = (u_1(x, y), u_2(x, y))$$

for all $(x, y) \in [n] \times [m]$.

For $u \in S([n] \times [m])$ let ${}^t u \in S([m] \times [n])$ be the transposed permutation defined by

$${}^t u(y, x) := (u_2(x, y), u_1(x, y)), \tag{1}$$

for all $(y, x) \in [m] \times [n]$. Notice that the diagram of ${}^t u$ is obtained by transposing that of u. Moreover, it is clear that it holds $u = {}^t({}^t u)$.

We note the following property of the transpose operation, whose verification is immediate from the definition.

Proposition 3.4. *Let $u, v \in S([n] \times [m])$ and $x \in S_n$, $y \in S_m$. Then*

(1) ${}^t(uv) = {}^t u \, {}^t v$;

(2) ${}^t(x \otimes y) = y \otimes x$.

In the rest of this work we denote by 1 the identity element of S_n (this should not cause confusion as n will always be clear from the context). So, for example, we write $1 \otimes 21$ instead of $12 \otimes 21$. Also, we find it often convenient to let, as in [9], if $u \in S([n]^r)$, $\varphi(u) := 1 \otimes u$, and to identify u with $u \otimes 1$. We call $\varphi(u)$ the *one-sided shift* of u.

4 Stable permutations

4.1 Definition

Throughout this section r is a natural number, $n \in \mathbb{N}$. We identify, as explained in the previous section, $S([n]^r)$ with S_{n^r}.

Let $u \in S([n]^r)$ and $k \in \mathbb{N}$. Define an element $\psi_k(u) \in S([n]^{r+k})$ by

$$\psi_k(u) := (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1})(\underbrace{1 \otimes \cdots \otimes 1}_{k-1} \otimes u^{-1} \otimes 1) \cdots$$
$$\cdots (1 \otimes u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-1})(u^{-1} \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k})(1 \otimes u \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-1})$$
$$(1 \otimes 1 \otimes u \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-2}) \cdots (\underbrace{1 \otimes \ldots \otimes 1}_{k-1} \otimes u \otimes 1)(\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u)$$

(and $\psi_0(u) := u^{-1}$). Equivalently,

$$\psi_k(u) = \prod_{i=0}^{k} (\underbrace{1 \otimes \ldots \otimes 1}_{k-i} \otimes u^{-1} \otimes \underbrace{1 \otimes \ldots \otimes 1}_{i}) \prod_{i=1}^{k} (\underbrace{1 \otimes 1 \ldots \otimes 1}_{i} \otimes u \otimes \underbrace{1 \otimes \ldots \otimes 1}_{k-i}) .$$

Note that

$$\psi_k(u) = (\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u^{-1})(\psi_{k-1}(u) \otimes 1)(\underbrace{1 \otimes \ldots \otimes 1}_{k} \otimes u)$$

for all $k \geq 1$.

Definition 4.1. Say that $u \in S([n]^r)$ is *stable* if there exists $k_0 \geq 0$ such that

$$\psi_k(u) = \psi_{k_0}(u) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k-k_0} \tag{2}$$

for all $k \geq k_0$. We let N_n^r denote the number of stable permutations in $S([n]^r)$.

For example, if $n = r = 2$, then it is not hard to check that there are four stable permutations, namely the identity and

$$((1,1),(2,1))((1,2),(2,2)),\ ((1,1),(1,2))((2,1),(2,2)),\ ((1,1),(2,2))((2,1),(1,2))$$

(that is, the three permutations listed in Example 3.3), so $N_2^2 = 4$. Also, for $r = 1$, all the elements in $S([n])$ are stable (they correspond to the so-called Bogolubov automorphisms of $\mathcal{O}_n$). Hence $N_n^1 = n!$. Note that the identity is stable, so N_n^r is positive for all n, r. Moreover, if $u \in S([n]^r)$ is stable, then it can be checked that $u \otimes 1 \in S([n]^{r+1})$ is stable. Therefore the stable permutations of $S([n]^r)$ are naturally embedded in the stable permutations of $S([n]^{r+1})$, so that $N_n^r \leq N_n^{r+1}$.

The next result follows from Theorem 2.1, taking into account the analysis carried out in [9] (see Theorem 2.2).

Theorem 4.2. *Let $n, r \in \mathbb{N}$, and $u \in S([n]^r)$. Then λ_u is an automorphism of $\mathcal{O}_n$ if and only if u is stable. In particular, the reduced Weyl group of $\mathcal{O}_n$ is*

$$\{\lambda_u : u \in S([n]^r), r \in \mathbb{N},\ u \ \textit{stable}\}.$$

4.2 The rank of a stable permutation

Say that a stable permutation has rank 1 if $k_0 = 0$, i.e. $\psi_k(u) = u^* \otimes \underbrace{1 \otimes \cdots \otimes 1}_{k}$ for all $k \geq 0$.

Definition 4.3. We say that a stable permutation has rank $k_0 + 1$ if k_0 is the smallest integer satisfying equation (2).

So if $u \in S([n]^2)$ then u has rank 1 if and only if $\psi_1(u) = u^{-1}$, u has rank ≤ 2 if and only if $\psi_2(u) = \psi_1(u)$, and in general u has rank $\leq k$ if and only if $\psi_k(u) = \psi_{k-1}(u)$. It is then easy to see that u has rank 1 if and only if u^{-1} has rank 1. For instance, all the permutations in $S([n])$ have rank 1. Furthermore, if $u \in S([n]^r)$ is stable then $\varphi(u)$ is also stable of the same rank. The three permutations listed in Example 3.3) all have rank 1.

Proposition 4.4. *Let $u \in S([n]^r)$, and $k \in \mathbb{N}$. Then*

i) if $\psi_k(u) \in S([n]^{k+1}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r-1}$ then u is stable of rank $\leq k+1$;

ii) if u is stable of rank $\leq k+1$ then $\psi_{k+1}(u) \in S([n]^{k+r}) \otimes 1$.

In particular, u is stable if and only if there exists a positive integer h such that $\psi_h(u) \in S([n]^{h+1}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r-1}$.

Proof. If $\psi_k(u) \in S([n]^{k+1}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r-1}$ then $\psi_h(u) = \psi_k(u) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{h-k}$ for all $h \geq k$ so u is stable of rank $\leq k+1$.

Conversely, if u is stable of rank $\leq k+1$ one has $\psi_{k+j}(u) = \psi_k(u) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{j}$ for all $j \geq 1$, so in particular $\psi_{k+1}(u) = \psi_k(u) \otimes 1$ and $\psi_k(u) \in S([n]^{r+k})$.

The last statement follows from the fact that if u is stable, say of rank $\leq k+1$, then, by what we have just observed, $\psi_{k+r-1}(u) \in S([n]^{r+k}) \otimes \underbrace{1 \otimes \cdots \otimes 1}_{r-1}$. ☐

4.3 Stable permutations of rank 1 (case $r = 2$)

Proposition 4.5. *Let $u \in S([n]^2)$. Then u is stable of rank 1 if and only if u satisfies the equation*

$$(u \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1) \tag{3}$$

in $S([n]^3)$.

In the shorthand notation, the above equation is written as $u\varphi(u) = \varphi(u)u$. From the point of view of the Cuntz algebras, the equation (3) says that λ_u is an automorphism of $\mathcal{O}_n$ and furthermore $(\lambda_u)^{-1} = \lambda_{u^{-1}}$. It would be interesting to know how many stable permutations of rank 1 there are in $S([n]^r)$.

Equation (3) somewhat resembles (but is different from) the Yang-Baxter equation (i.e., $u\varphi(u)u = \varphi(u)u\varphi(u)$, see [26] for a general introduction). Indeed, if a permutation $u \in S([n]^2)$ satisfies both equation (3) and the Yang-Baxter equation then $u = \varphi(u)$ so $u = 1$. After Drinfel'd's suggestion [18], permutations $u \in S([n]^2)$ that satisfy the Yang-Baxter equation have been widely studied (see, e.g., [3, 19, 21, 20, 23, 27]) and are often called *set-theoretic solutions* of the YBE. More generally, it can be shown (see [5]) that any set-theoretic solution $u \neq 1$ of the YBE is not a stable permutation.

Remark 4.6. Note that it is not true that if $u \in S([n]^2)$ satisfies equation (3) and $\sigma, \tau \in S_n$ then $u^{\sigma\otimes\tau}$ satisfies equation (3), where $u^{\sigma\otimes\tau} := (\sigma^{-1}\otimes\tau^{-1})u(\sigma\otimes\tau)$ $(= \sigma^{-1}\varphi(\tau^{-1})u\sigma\varphi(\tau))$. For example, if $n = 3$, $r = 2$, $u = 132456789$, $\sigma = 321$ and $\tau = 213$ then $u^{\sigma\otimes\tau} = 123456987$ which does not satisfy equation (3). Furthermore, it is not true that if $u, v \in S([n]^2)$ both satisfy equation (3) then uv satisfies equation (3). For example, if $u = 132456789$ and $v = 126453789$ then $vu = 162453789$ which does not satisfy equation (3).

Remark 4.7. Let $u \in S([n]^2)$ be a solution of the equation (3). Then, u being stable, for any $z \in S_n$ one has that $zu\varphi(z)^*$ is also stable (see [9], and also Proposition 6.2 below). However, it is not generally true that $zu\varphi(z)^*$ still satisfies equation (3). Indeed, this amounts to checking that $zu\varphi(z^*)\varphi(zu\varphi(z^*)) =$

$\varphi(zu\varphi(z^*))zu\varphi(z^*)$, that is

$$zu\varphi(u)\varphi^2(z^*) = \varphi(z)\varphi(u)\varphi^2(z^*)zu\varphi(z^*).$$

It is easy to see that $\varphi^2(z^*)$ commutes with $zu\varphi(z^*)$ and that z commutes with $\varphi(z)\varphi(u)$. After simplifying, one is thus left with the condition

$$u\varphi(u) = \varphi(z)\varphi(u)u\varphi(z^*) \ ,$$

that is $u\varphi(u)$ $(= \varphi(u)u)$ commutes with $\varphi(z)$.

If one also knows that u is an involution then $zu\varphi(z)^*$ is an involution if and only if $zu\varphi(z^*)zu\varphi(z^*) = 1$, i.e. $zuz = \varphi(z)u\varphi(z)$.

We find it convenient to have an explicit characterization of permutations of rank 1. We provide the following more general result, that will be used often in the sequel. We use the maps u_i and u_2 introduced in Section 3.

Proposition 4.8. *Let* $u, v \in S([n]^2)$. *Then* $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$ *if and only if*

$$\begin{aligned} v_1(\alpha,\beta) &= v_1(\alpha, u_1(\beta,\gamma)) \\ u_1(v_2(\alpha,\beta),\gamma) &= v_2(\alpha, u_1(\beta,\gamma)) \\ u_2(v_2(\alpha,\beta),\gamma) &= u_2(\beta,\gamma) \end{aligned}$$

for all $\alpha, \beta, \gamma \in [n]$.

Proof. If $\alpha, \beta, \gamma \in [n]$, we have that

$$\begin{aligned} &(1 \otimes u)(v \otimes 1)((\alpha-1)n^2 + (\beta-1)n + \gamma) \\ &\qquad = (1 \otimes u)(v \otimes 1)([(\alpha-1)n + (\beta-1)]n + \gamma) \\ &\qquad = (1 \otimes u)([v((\alpha-1)n + \beta) - 1]n + \gamma) \\ &\qquad = (1 \otimes u)([(v_1(\alpha,\beta)-1)n + v_2(\alpha,\beta) - 1]n + \gamma) \\ &\qquad = (1 \otimes u)((v_1(\alpha,\beta)-1)n^2 + (v_2(\alpha,\beta)-1)n + \gamma) \\ &\qquad = (v_1(\alpha,\beta)-1)n^2 + u((v_2(\alpha,\beta)-1)n + \gamma) \\ &\qquad = (v_1(\alpha,\beta)-1)n^2 + (u_1(v_2(\alpha,\beta),\gamma)-1)n + u_2(v_2(\alpha,\beta),\gamma) \ . \end{aligned}$$

Similarly,

$$\begin{aligned}
(v \otimes 1)(1 \otimes u)((\alpha-1)n^2 + (\beta-1)n + \gamma) \\
&= (v \otimes 1)((\alpha-1)n^2 + u((\beta-1)n+\gamma)) \\
&= (v \otimes 1)((\alpha-1)n^2 + (u_1(\beta,\gamma)-1)n + u_2(\beta,\gamma)) \\
&= (v \otimes 1)([(\alpha-1)n + u_1(\beta,\gamma)-1]n + u_2(\beta,\gamma)) \\
&= [v((\alpha-1)n + u_1(\beta,\gamma)) - 1]n + u_2(\beta,\gamma) \\
&= [(v_1(\alpha, u_1(\beta,\gamma))-1)n + v_2(\alpha, u_1(\beta,\gamma)) - 1]n + u_2(\beta,\gamma) \\
&= (v_1(\alpha, u_1(\beta,\gamma))-1)n^2 + (v_2(\alpha, u_1(\beta,\gamma))-1)n + u_2(\beta,\gamma) \ .
\end{aligned}$$

□

Corollary 4.9. *Let $u, v \in S([n]^2)$ be such that $u_1(x,y) = x$ and $v_2(x,y) = y$ for all $x, y \in [n]$. Then $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$.*

Let $u, v \in S([n]^2)$. Then we have that, if $\alpha, \beta \in [n]$,

$$\begin{aligned}
(vu)((\alpha-1)n+\beta) &= v((u_1(\alpha,\beta)-1) + u_2(\alpha,\beta)) \\
&= (v_1(u_1(\alpha,\beta), u_2(\alpha,\beta)) - 1)n + v_2(u_1(\alpha,\beta), u_2(\alpha,\beta)) \ .
\end{aligned}$$

Therefore, $uv = vu$ if and only if

$$\begin{aligned}
v_1(u_1(\alpha,\beta), u_2(\alpha,\beta)) &= u_1(v_1(\alpha,\beta), v_2(\alpha,\beta)) \\
v_2(u_1(\alpha,\beta), u_2(\alpha,\beta)) &= u_2(v_1(\alpha,\beta), v_2(\alpha,\beta))
\end{aligned}$$

for all $\alpha, \beta \in [n]$. In particular, $v = u^{-1}$ if and only if

$$\begin{aligned}
v_1(u_1(\alpha,\beta), u_2(\alpha,\beta)) &= \alpha \\
v_2(u_1(\alpha,\beta), u_2(\alpha,\beta)) &= \beta
\end{aligned}$$

for all $\alpha, \beta \in [n]$.

Let $x, y, w, z \in S_n$, $u := x \otimes y$, $v := w \otimes z$. Then, by the definitions, $u_1(\alpha,\beta) = x(\alpha)$, $u_2(\alpha,\beta) = y(\beta)$, $v_1(\alpha,\beta) = w(\alpha)$, $v_2(\alpha,\beta) = z(\beta)$ for all $\alpha, \beta \in [n]$. Therefore, we conclude that $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$ if and only if $xz = zx$ (in S_n). The following proposition is almost immediate.

Proposition 4.10. *Let $u \in S([n]^2)$, $u = x \otimes y$, $x, y \in S_n$. Then u is stable of rank 1 if and only if $xy = yx$.*

4.4 Stable permutations of rank two ($r = 2$)

Let $u \in S([n]^2)$. Consider the following equation in $S([n]^3)$

$$(u^{-1} \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1). \tag{4}$$

This says that $\psi_1(u) = u$ so $\psi_k(u) = u$ for all $k \geq 1$, hence u is stable of rank ≤ 2. From the point of view of the Cuntz algebras, it says that λ_u is an automorphism of $\mathcal{O}_n$ such that $(\lambda_u)^{-1} = \lambda_u$, that is λ_u is an involution. Indeed, the converse statement is also true, as the following result shows.

Proposition 4.11. *Let $u \in S([n]^2)$. Then u satisfies the equation (4) if and only if $(\lambda_u)^{-1} = \lambda_u$. Moreover, in this case u is stable of rank ≤ 2.*

Proof. We have that $(\lambda_u)^2 = \lambda_{\lambda_u(u)u}$. But, if $u \in S([n]^2)$, then $\lambda_u(u) = u\varphi(u)u\varphi(u^{-1})u^{-1}$, so $\lambda_{\lambda_u(u)u} = \lambda_{u\varphi(u)u\varphi(u^{-1})}$. Hence $(\lambda_u)^2 = Id$ if and only if $\lambda_{u\varphi(u)u\varphi(u^{-1})} = Id = \lambda_1$ which happens if and only if $u\varphi(u)u\varphi(u^{-1}) = 1$. □

Proposition 4.12. *Let $u \in S([n]^2)$, $u = x \otimes y$, $x, y \in S_n$. Then*

$$(u^{-1} \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1)$$

if and only if $x^{-1} = x$ and $yx = (yx)^{-1}$.

Let $u \in S([n]^2)$. We consider more generally the following condition

$$(1 \otimes u^{-1})(u^{-1} \otimes 1)(1 \otimes u) \in S([n]^2) \otimes 1. \tag{5}$$

As above, such permutations are stable of rank ≤ 2. From the point of view of the Cuntz algebras, equation (5) says that λ_u is an automorphism of $\mathcal{O}_n$ such that $(\lambda_u)^{-1} = \lambda_v$ for a certain $v \in S([n]^2)$.

In general, if $u \in S([n]^2)$, one might consider the family of conditions

$$\psi_k(u) \in S([n]^{k+1}) \otimes 1$$

for $k \geq 2$. Such permutations are stable of rank $\leq k + 1$ (cf. Proposition 4.4).

Remark 4.13. Suppose that u, v are stable permutations in $S([n]^2)$ (not necessarily of rank 1). Then the commutativity $\lambda_u\lambda_v = \lambda_v\lambda_u$ of the associated automorphisms of $\mathcal{O}_n$ is equivalent to the validity of the relation

$$(u \otimes 1)(1 \otimes u)(v \otimes 1)(1 \otimes u^{-1}) = (v \otimes 1)(1 \otimes v)(u \otimes 1)(1 \otimes v^{-1}) \tag{6}$$

in $S([n]^3)$.

The following result establishes a connection between rank 1 and rank two.

Proposition 4.14. *Let $u \in S([n]^2)$. If $\psi_1(u) := \varphi(u)^* u^* \varphi(u) \in S([n]^3)$ is stable of rank ≤ 1 then u is stable of rank ≤ 2.*

Proof. By assumption, we have $\psi_1(\psi_1(u)) = \psi_1(u)^*$, that is

$$\varphi\big(\varphi(u)^* u \varphi(u)\big)\varphi(u)^* u \varphi(u) \varphi\big(\varphi(u)^* u^* \varphi(u)\big) = \varphi(u^*) u \varphi(u) \ .$$

Moreover, $\psi_2(\psi_1(u)) = \psi_1(u)^*$, that is

$$\begin{aligned}\varphi^2\big(\varphi(u)^* u \varphi(u)\big)\varphi\big(\varphi(u)^* u \varphi(u)\big)\varphi(u)^* u \varphi(u)\varphi\big(\varphi(u)^* u^* \varphi(u)\big)\varphi^2\big(\varphi(u)^* u^* \varphi(u)\big) \\ = \varphi(u^*) u \varphi(u)\end{aligned}$$

and thus

$$\varphi^2\big(\varphi(u)^* u \varphi(u)\big)\varphi(u^*) u \varphi(u) \varphi^2\big(\varphi(u)^* u^* \varphi(u)\big) = \varphi(u^*) u \varphi(u) \ .$$

Therefore, after simplifying the highest powers, we get

$$\varphi^2(u)\varphi(u^*) u \varphi(u) \varphi^2(u^*) = \varphi(u^*) u \varphi(u) \ ,$$

which, after taking the adjoints of both sides, can be written as

$$\varphi^2(u^*)\varphi(u^*) u^* \varphi(u) \varphi^2(u) = \varphi(u^*) u^* \varphi(u)$$

which means that $\psi_2(u) = \psi_1(u)$. It then follows at once that $\psi_k(u) = \psi_1(u)$ for all $k \geq 2$, and u is stable of rank two. □

Note that the argument works equally well for unitary matrices [9]. One might wonder whether a similar claim holds for $u \in \mathcal{U}(M_{n^r})$, with $r > 2$.

4.5 An arithmetical digression

If we identify $[n]^3$ with $[n^3]$ as explained in Section 3 then the above equations lead to seemingly complicated arithmetical questions. In this subsection we spell out in detail one such instance, namely the one arising from equation (4).

We first discuss a change of basis formula, relating quotients and remainders modulo n and n^2, respectively.

Proposition 4.15. *Let $j \in [n^3]$. We can write $j = n(i_1 - 1) + k_1$ with $i_1 \in [n^2]$, $k_1 \in [n]$ and $j = n^2(i_2 - 1) + k_2$ with $i_2 \in [n]$ and $k_2 \in [n^2]$. Then we have*

$$i_1 = n(i_2 - 1) + \Big\lfloor \frac{k_2 - 1}{n} \Big\rfloor + 1, \qquad k_1 = k_2 - n \Big\lfloor \frac{k_2 - 1}{n} \Big\rfloor$$

$$i_2 = \Big\lfloor \frac{i_1 - 1}{n} \Big\rfloor + 1, \qquad k_2 = n^2 \Big\{ \frac{i_1 - 1}{n} \Big\} + k_1.$$

Proof. Let $j = n^2(\alpha-1)+n(\beta-1)+\gamma$, $\alpha,\beta,\gamma \in [n]$. Now, from $n(i_1-1)+k_1 = n^2(\alpha-1)+n(\beta-1)+\gamma$ get $k_1 = \gamma$ and $i_1 - 1 = n(\alpha-1)+(\beta-1)$ and from $n^2(i_2-1)+k_2 = n^2(\alpha-1)+n(\beta-1)+\gamma$ get $k_2 = n(\beta-1)+\gamma$ and $i_2 = \alpha$. Therefore, $k_1 = \gamma = k_2 - n(\beta-1)$ and thus $\beta - 1 = \frac{k_2-\gamma}{n} = \frac{k_2-1}{n} + \frac{1-\gamma}{n}$. Hence, $\frac{k_2-1}{n} = \beta - 1 + \frac{\gamma-1}{n}$ and

$$\beta - 1 = \Big\lfloor \frac{k_2-1}{n} \Big\rfloor \ .$$

We have thus obtained that

$$i_1 = n(i_2-1) + \Big\lfloor \frac{k_2-1}{n} \Big\rfloor + 1 \ , \qquad k_1 = k_2 - n\Big\lfloor \frac{k_2-1}{n} \Big\rfloor \ .$$

On the other hand, from $\beta - 1 = (i_1-1) - n(\alpha-1)$ we deduce that $k_2 = n(\beta-1)+\gamma = n[(i_1-1)-n(\alpha-1)] + k_1 = n[(i_1-1)-n(i_2-1)] + k_1$. Since $i_2 = \alpha = \frac{i_1-\beta}{n} + 1 = \frac{i_1-1}{n} + \frac{1-\beta}{n} + 1$ we get $i_2 - 1 = \frac{i_1-1}{n} + \frac{1-\beta}{n}$, that is $\frac{i_1-1}{n} = i_2 - 1 + \frac{\beta-1}{n}$ so that $\Big\lfloor \frac{i_1-1}{n} \Big\rfloor = i_2 - 1$ and so $k_2 = n[(i_1-1) - n\Big\lfloor \frac{i_1-1}{n} \Big\rfloor] + k_1$. Therefore,

$$i_2 = \Big\lfloor \frac{i_1-1}{n} \Big\rfloor + 1 \ , \qquad k_2 = n^2\Big\{ \frac{i_1-1}{n} \Big\} + k_1 \ .$$

(Notice that k_1 depends only on k_2 and i_2 depends only on i_1.) $\square$

For $u \in S([n]^2)$ one has

$$(u \otimes 1)(n(i_1-1)+k_1) = n(u(i_1)-1)+k_1$$

and

$$(1 \otimes u)(n^2(i_2-1)+k_2) = n^2(i_2-1)+u(k_2) \ .$$

For $j \in [n^3]$ we compute

$$\begin{aligned}
(1\otimes u)(u\otimes 1)(j) &= (1\otimes u)(u\otimes 1)(n(i_1-1)+k_1) \\
&= (1\otimes u)(n(u(i_1)-1)+k_1) \\
&= (1\otimes u)\Big(n^2\Big[\Big(\Big\lfloor \frac{u(i_1)-1}{n} \Big\rfloor + 1\Big) - 1\Big] + n^2\Big\{\frac{u(i_1)-1}{n}\Big\} + k_1\Big) \\
&= n^2\Big[\Big(\Big\lfloor \frac{u(i_1)-1}{n} \Big\rfloor + 1\Big) - 1\Big] + u\Big(n^2\Big\{\frac{u(i_1)-1}{n}\Big\} + k_1\Big) \\
&= n^2\Big\lfloor \frac{u(i_1)-1}{n} \Big\rfloor + u\Big(n^2\Big\{\frac{u(i_1)-1}{n}\Big\} + k_1\Big) \\
&= n^2\Big\lfloor \frac{u(n(i_2-1)+\lfloor\frac{k_2-1}{n}\rfloor+1)-1}{n} \Big\rfloor \\
&\qquad + u\Big(n^2\Big\{\frac{u(n(i_2-1)+\lfloor\frac{k_2-1}{n}\rfloor+1)-1}{n}\Big\} + k_2 - n\Big\lfloor\frac{k_2-1}{n}\Big\rfloor\Big)
\end{aligned}$$

and

$$
\begin{aligned}
(u^{-1}\otimes 1)(1\otimes u)(j) &= (u^{-1}\otimes 1)(1\otimes u)(n^2(i_2-1)+k_2)\\
&= (u^{-1}\otimes 1)(n^2(i_2-1)+u(k_2))\\
&= (u^{-1}\otimes 1)\Big(n\Big[\Big(n(i_2-1)+\Big\lfloor\frac{u(k_2)-1}{n}\Big\rfloor+1\Big)-1\Big]\\
&\qquad +u(k_2)-n\Big\lfloor\frac{u(k_2)-1}{n}\Big\rfloor\Big)\\
&= n\Big[u^{-1}\Big(n(i_2-1)+\Big\lfloor\frac{u(k_2)-1}{n}\Big\rfloor+1\Big)-1\Big]\\
&\qquad +u(k_2)-n\Big\lfloor\frac{u(k_2)-1}{n}\Big\rfloor
\end{aligned}
$$

5 Compatibility

It is not hard to show, using the connection between stable permutations and automorphisms of the Cuntz algebra, that if $w,z\in S([n]^2)$ are stable, and $z\varphi(w)=\varphi(w)z$, then wz is also stable. We give a combinatorial proof of this fact here, which is self contained and also has the advantage of giving more information on the product. We need first the following preliminary identity.

Lemma 5.1. *Let $w,z\in S([n]^2)$ be such that $z\varphi(w)=\varphi(w)z$. Then, for every $k\geq 1$,*

$$\psi_k(wz)=\varphi^k(z)^*\varphi^{k-1}(z)^*\ldots\varphi(z)^*z^*\psi_k(w)\varphi(z)\ldots\varphi^{k-1}(z)\varphi^k(z)\ .$$

Proof. We show the claim by induction on k. For $k=1$ we have

$$
\begin{aligned}
\psi_1(wz) &= \varphi(wz)^*(wz)^*\varphi(wz)=\varphi(z)^*\varphi(w)^*z^*w^*\varphi(w)\varphi(z)\\
&= \varphi(z)^*z^*\varphi(w)^*w^*\varphi(w)\varphi(z)=\varphi(z)^*z^*\psi_1(w)\varphi(z)\ .
\end{aligned}
$$

Now, let $k\geq 2$. Then by the induction hypothesis and the commutation rule we have

$$
\begin{aligned}
\psi_k(wz) &= \varphi^k(wz)^*\psi_{k-1}(wz)\varphi^k(wz)\\
&= \varphi^k(z)^*\varphi^k(w)^*\big(\varphi^{k-1}(z)^*\ldots\varphi(z)^*z^*\psi_{k-1}(w)\varphi(z)\ldots\varphi^{k-1}(z)\big)\varphi^k(w)\varphi^k(z)\\
&= \varphi^k(z)^*\varphi^{k-1}(z)^*\ldots\varphi(z)^*z^*\varphi^k(w)^*\psi_{k-1}(w)\varphi^k(w)\varphi(z)\ldots\varphi^{k-1}(z)\varphi^k(z)
\end{aligned}
$$

as desired. □

We can now prove that the product of two stable permutations as above (is stable and) has rank bounded above by the sum of the ranks of the factors.

Theorem 5.2. *Let $w, z \in S([n]^2)$ be such that $z\varphi(w) = \varphi(w)z$. Assume also that w is stable of rank $\leq r$ and z is stable of rank $\leq s$. Then wz is stable of rank $\leq r + s$.*

Proof. For $s = 1$ and any r the statement can be checked directly without much trouble. We shall henceforth assume that $s \geq 2$.
We claim that $\psi_{r+s-1}(wz)$ equals

$$\psi_{s-1}(z)\varphi(\psi_{s-1}(z))\ldots\varphi^r(\psi_{s-1}(z))\psi_{r-1}(w)\varphi^r(\psi_{s-1}(z))^*\ldots\varphi(\psi_{s-1}(z))^* \ . \quad (7)$$

We begin by showing that

$$\psi_{s-1}(z)\varphi(\psi_{s-1}(z))\ldots\varphi^r(\psi_{s-1}(z)) = \varphi^{r+s-1}(z)^*\ldots\varphi(z)^*z^*\varphi^{r+1}(z)\ldots\varphi^{r+s-1}(z) \ . \quad (8)$$

(In this identity, r can actually be *any* nonnegative integer.) Note that, since z is stable of rank $\leq s$ we have that

$$\psi_{s-1}(z)\varphi(\psi_{s-1}(z))\ldots\varphi^r(\psi_{s-1}(z)) = \psi_{r+s-1}(z)\varphi(\psi_{r+s-2}(z))\ldots\varphi^r(\psi_{s-1}(z)) \ .$$

We claim that

$$\begin{aligned}&\psi_{r+s-1}(z)\varphi(\psi_{r+s-2}(z))\ldots\varphi^{k-1}(\psi_{r+s-k}(z))\\ &\qquad = \varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\varphi^k(z)\cdots\varphi^{r+s-1}(z)\end{aligned}$$

for all $k = 1, \ldots, r+1$. We prove this claim by induction on k, the identity being true by definition if $k = 1$. If $k \geq 2$ then we have by induction that

$$\begin{aligned}&\psi_{r+s-1}(z)\varphi(\psi_{r+s-2}(z))\ldots\varphi^{k-1}(\psi_{r+s-k}(z))\\ &= \psi_{r+s-1}(z)\varphi(\psi_{r+s-2}(z))\ldots\varphi^{k-2}(\psi_{r+s-k+1}(z))\varphi^{k-1}(\psi_{r+s-k}(z))\\ &= \varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\varphi^{k-1}(z)\cdots\varphi^{r+s-1}(z)\varphi^{k-1}(\psi_{r+s-k}(z))\\ &= \varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\varphi^{k-1}(z)\cdots\varphi^{r+s-1}(z)\\ &\qquad \times \varphi^{k-1}\big(\varphi^{r+s-k}(z)^*\cdots\varphi(z)^*z^*\varphi(z)\cdots\varphi^{r+s-k}(z)\big)\end{aligned}$$

as claimed. This proves (8). Now, substituting (8) into (7) we obtain

$$\begin{aligned}
&\varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\varphi^{r+1}(z)\cdots\varphi^{r+s-1}(z)\psi_{r-1}(w)\\
&\qquad\times\varphi^r(\psi_{s-1}(z))^*\ldots\varphi(\psi_{s-1}(z))^*\\
&=\varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\varphi^{r+1}(z)\cdots\varphi^{r+s-1}(z)\psi_{r-1}(w)\\
&\qquad\times\varphi\big(\varphi^{r-1}(\psi_{s-1}(z))^*\ldots\psi_{s-1}(z)^*\big)\\
&=\varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\varphi^{r+1}(z)\cdots\varphi^{r+s-1}(z)\psi_{r-1}(w)\\
&\qquad\times\varphi\big(\varphi^{r+s-2}(z)^*\ldots\varphi^{r}(z)^*z\varphi(z)\ldots\varphi^{r+s-2}(z)\big)\\
&=\varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\psi_{r-1}(w)\varphi(z)\varphi^2(z)\ldots\varphi^{r+s-1}(z)\\
&=\varphi^{r+s-1}(z)^*\varphi^{r+s-2}(z)^*\cdots\varphi(z)^*z^*\psi_{r+s-1}(w)\varphi(z)\varphi^2(z)\ldots\varphi^{r+s-1}(z)\\
&=\psi_{r+s-1}(wz)
\end{aligned}$$

by Lemma 5.1 and our hypothesis that w is stable of rank $\leq r$. This proves our claim. Analogously, we have that

$$\psi_{r+s}(wz)=\psi_s(z)\varphi(\psi_s(z))\ldots\varphi^r(\psi_s(z))\psi_{r-1}(w)\varphi^r(\psi_s(z))^*\ldots\varphi(\psi_s(z))^*$$

Since $\psi_s(z)=\psi_{s-1}(z)$ we conclude that $\psi_{r+s}(wz)=\psi_{r+s-1}(wz)$. □

The previous result has a sort of partial converse, as we now show.

Proposition 5.3. *Let $w, z \in S([n]^2)$ be such that $z\varphi(w)=\varphi(w)z$. Assume that w is stable and z is not stable. Then wz is not stable.*

Proof. We have, since $z \in S([n]^2)$ and $z\varphi(w)=\varphi(w)z$, that $\lambda_w\lambda_z=\lambda_{\lambda_w(z)w}=\lambda_{w\varphi(w)z\varphi(w)^{-1}}=\lambda_{wz}$. But, by Theorem 4.2 and our hypothesis, λ_z is not an automorphism, so λ_{wz} is not an automorphism, and the result follows again from Theorem 4.2. □

Because of the previous results, we say that two permutations $u, w \in S([n]^2)$ are *compatible* if $(1\otimes u)(w\otimes 1)=(w\otimes 1)(1\otimes u)$. Note that it is not true, in general, that if u, w are compatible then w, u are compatible.

Corollary 5.4. *Let $u_1, u_2, \ldots, u_m \in S([n]^2)$ be stable such that $(u_j\otimes 1)(1\otimes u_i)=(1\otimes u_i)(u_j\otimes 1)$ for all $1\leq i<j\leq m$. Then $u_1u_2\cdots u_m \in S([n]^2)$ is stable.*

Proof. For $m=2$ the result follows from Theorem 5.2. If $m>2$ then by induction we have that $u_1\cdots u_{m-1}\in S([n]^2)$ is stable. But

$$\begin{aligned}
(u_m\otimes 1)(1\otimes u_1\cdots u_{m-1})&=(u_m\otimes 1)(1\otimes u_1)\cdots(1\otimes u_{m-1})\\
&=(1\otimes u_1)\cdots(1\otimes u_{m-1})(u_m\otimes 1)\\
&=(1\otimes u_1\cdots u_{m-1})(u_m\otimes 1)
\end{aligned}$$

so $u_1\cdots u_m\in S([n]^2)$ is also stable. □

It is worth stressing that, by Theorem 5.2, the product $u_1 \dots u_m$ of m stable permutations of rank 1 u_i in $S([n]^2)$ that is compatible in at least one way, i.e., for which there exists a bracketing such that each properly parenthesized subword is (stable and) compatible with its sibling, is stable and has rank bounded by m. In this case, we will call $u_1 \dots u_m$ a *compatible product*.

It is of interest to explicitly identify specific subgroups of $\mathrm{Aut}(\mathcal{O}_n)$. In this respect the previous result has the following consequence.

Corollary 5.5. *Let $u_1, u_2, \dots, u_m \in S([n]^2)$ be stable permutations of rank 1 such that $(u_j \otimes 1)(1 \otimes u_i) = (1 \otimes u_i)(u_j \otimes 1)$ for all $1 \le i, j \le m$. Then the subgroup of $\mathrm{Aut}(\mathcal{O}_n)$ generated by $\{\lambda_{u_1}, \dots, \lambda_{u_m}\}$ is isomorphic to the subgroup of $S([n]^2)$ generated by $\{u_1, \dots, u_m\}$.*

In view of Theorem 5.2 (and Theorem 8.1 below), it is an interesting problem to decide when two (stable) transpositions of $S([n]^2)$ are compatible. In this direction we have the following result.

Proposition 5.6. *Let $u = ((a, b), (i, j))$, where $a, b, i, j \in [n]$, $b \neq j$, and $v \in S([n]^2)$. Then $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$ if and only if there is $\sigma \in S_n$ such that $v(x, k) = (\sigma(x), k)$ for all $x \in [n]$ and all $k \in \{a, i\}$.*

Proof. Note first that

$$1 \otimes u = \prod_{x=1}^{n} ((x, a, b), (x, i, j)),$$

and hence that

$$(v \otimes 1)(1 \otimes u)(v^{-1} \otimes 1) = \prod_{x=1}^{n} ((v(x, a), b), (v(x, i), j)).$$

Therefore u is compatible with v if and only if these two permutations are the same. But this happens if and only if there is a permutation $\sigma \in S_n$ such that $((v(x, a), b), (v(x, i), j)) = ((\sigma(x), a, b), (\sigma(x), i, j))$ for all $x \in [n]$, so if and only if (since $b \neq j$) there is a permutation $\sigma \in S_n$ such that $v(x, a) = (\sigma(x), a)$ and $v(x, i) = (\sigma(x), i)$ for all $x \in [n]$, as claimed. $\square$

The previous result covers all transpositions except the "vertical" ones. For these the situation is slightly more involved, as we now show.

Proposition 5.7. *Let $u = ((a, b), (i, b))$, where $a, b, i \in [n]$, $a \neq i$, and $v \in S([n]^2)$. Then $(v \otimes 1)(1 \otimes u) = (1 \otimes u)(v \otimes 1)$ if and only if there are $\sigma \in S_n$ and $\varepsilon \in (S_2)^n$ such that, for all $x \in [n]$,*

$$v(x, a) = (\sigma(x), a), \quad v(x, i) = (\sigma(x), i)$$

if $\varepsilon_x = \mathrm{Id}$*, while*

$$v(x,a) = (\sigma(x), i), \quad v(x,i) = (\sigma(x), a)$$

if $\varepsilon_x \neq \mathrm{Id}$.

Proof. We proceed as in the previous proof. We have that

$$1 \otimes u = \prod_{x=1}^{n} ((x,a,b),(x,i,b)),$$

and

$$(v \otimes 1)(1 \otimes u)(v^{-1} \otimes 1) = \prod_{x=1}^{n} ((v(x,a),b),(v(x,i),b)).$$

Therefore u is compatible with v if and only if these two permutations are the same. But this happens if and only if there is a permutation $\sigma \in S_n$ such that $((v(x,a),b),(v(x,i),b)) = ((\sigma(x),a,b),(\sigma(x),i,b))$ for all $x \in [n]$, so if and only if there is a permutation $\sigma \in S_n$ and an $\varepsilon \in (S_2)^n$ such that

$$[(v(x,a),b),(v(x,i),b)] = \begin{cases} [(\sigma(x),a,b),(\sigma(x),i,b)], & \text{if } \varepsilon = \mathrm{Id}, \\ [(\sigma(x),i,b),(\sigma(x),a,b)], & \text{if } \varepsilon \neq \mathrm{Id}, \end{cases}$$

for all $x \in [n]$, and the result follows. □

In particular, Proposition 5.6 shows that Corollary 5.5 applies to "horizontal" stable transpositions that act on the same row of $[n]^2$, but fix the diagonal. Namely, if $h,i,j,k,l \in [n]$, l,j,k distinct, l,h,i distinct, then $((l,h),(l,i))$ and $((l,j),(l,k)) \in S([n]^2)$ are compatible, in both orders.

Corollary 5.8. *Let* $u \in S([n]^2)$ *be such that* u *permutes the off-diagonal elements of a row. Then* u *is stable.*

Proof. Let l be the row that is permuted by u. Then u is a product of transpositions of the form $((l,i)(l,j))$ where $i,j \in [n] \setminus \{l\}$. But, as observed above, all these transpositions are mutually compatible by Proposition 5.6, and they are all stable of rank 1 again by Proposition 5.6 (applied to $u = v$). So the result follows from Corollary 5.4. □

Note that, by Proposition 5.13, the last two statements hold, respectively, for "vertical" stable transpositions of any column, and for permutations that permute the off-diagonal elements of any column.

From the point of view of $\mathrm{Aut}(\mathcal{O}_n)$, the previous result implies that the reduced Weyl group of $\mathcal{O}_n$, at level 2, contains several isomorphic copies of the Weyl group of type A of rank $n-2$.

Corollary 5.9. *$\{\lambda_u : u \in S([n]^2), u$ stable$\}$ contains at least $2n$ subgroups of $\mathrm{Aut}(\mathcal{O}_n)$ isomorphic to S_{n-1}, which pairwise intersect only at the identity.*

Proof. Let $l \in [n]$. By Proposition 5.6 any two transpositions that switch two elements of the l-th row of $[n]^2$, that are not on the diagonal, are compatible. Therefore, by Corollary 5.5, the subgroup of $\mathrm{Aut}(\mathcal{O}_n)$ generated by the automorphisms indexed by these transpositions is isomorphic to the subgroup R_l of $S([n]^2)$ generated by the transpositions themselves, which is easily seen to be isomorphic to S_{n-1}. Similarly for the l-th column and the corresponding subgroup C_l. Let now $i, j \in [n]$, and $u \in R_i \cap R_j$. Then $u(x, y) = (x, y)$ if $x \neq i$ and if $x \neq j$, so $u = Id_{[n]^2}$ if $i \neq j$. Similarly if $u \in C_i \cap C_j$. Finally, if $u \in R_i \cap C_j$ then $u(x, y) = (x, y)$ if $x \neq i$ and if $y \neq j$, so necessarily $u = Id_{[n]^2}$. □

We present yet another consequence of Proposition 5.6.

Proposition 5.10. *Let $u, v \in S([n]^2)$ be given by $u := ((i, j)(i, k))$, $v = ((a, c)(b, c))$, where $j \neq k$, $a \neq b$ ($i, j, k, a, b, c \in [n]$). Then $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$. In particular, uv is stable if $i \notin \{j, k\}$ and $c \notin \{a, b\}$.*

Note that the product of a vertical stable transposition with a horizontal one (in this order) is not necessarily stable; for example, $((1, 2), (3, 2))((1, 2), (1, 3))$ is not stable.

Given the importance of the compatibility condition we now investigate some further properties of this operation.

For $u \in S([n]^2)$ and $\sigma \in S_n$ define $u^\sigma := (\sigma^{-1} \otimes \sigma^{-1}) u (\sigma \otimes \sigma)$. So $u^\sigma \in S([n]^2)$ and

$$u^\sigma(x, y) = \big(\sigma^{-1}(u_1(\sigma(x), \sigma(y))), \sigma^{-1}(u_2(\sigma(x), \sigma(y)))\big)$$

for all $(x, y) \in [n]^2$.

Proposition 5.11. *Let $u, v \in S([n]^2)$ and let $\sigma \in S_n$. Then*

$$(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$$

if and only if

$$(1 \otimes u^\sigma)(v^\sigma \otimes 1) = (v^\sigma \otimes 1)(1 \otimes u^\sigma)\ .$$

Proof. This follows immediately from the observation that

$$(1 \otimes u^\sigma) = (\sigma^{-1} \otimes \sigma^{-1} \otimes \sigma^{-1})(1 \otimes u)(\sigma \otimes \sigma \otimes \sigma)$$

and similarly for v. □

Remark 5.12. As usual, let $F \in S([n]^2)$ denote the involutive permutation corresponding to the flip map, i.e., $F(x,y) = (y,x)$ for all $(x,y) \in [n]^2$. If $u \in S([n]^2)$, one has that u solves Equation (3) if and only if FuF does. Indeed, assuming the first relation, the second one can be quickly checked by means of the identity $\varphi(u) = F\varphi(F)u\varphi(F)F$ and repeated applications of the Yang- Baxter equation for F: first observe that the equality $F\varphi(F)u\varphi(F)Fu = uF\varphi(F)u\varphi(F)F$ implies that $u\varphi(F)FuF\varphi(F) = \varphi(F)FuF\varphi(F)u$, and then

$$\begin{aligned} FuF\varphi(F)\varphi(u)\varphi(F) &= FuF\varphi(F)F\varphi(F)u\varphi(F)F\varphi(F) \\ &= Fu\varphi(F)FuF\varphi(F)F \\ &= F\varphi(F)FuF\varphi(F)uF \\ &= \varphi(F)F\varphi(F)u\varphi(F)F\varphi(F)FuF \\ &= \varphi(F)\varphi(u)\varphi(F)FuF\ . \end{aligned}$$

Note that FuF is nothing but the transposed of u.

For $u \in S([n]^2)$, let ${}^t u, {}^a u, {}^\pi u \in S([n]^2)$ be the transposed, antitransposed and rotated (by π) permutations defined by

$${}^t u(x,y) := (u_2(y,x), u_1(y,x)), \tag{9}$$

$${}^a u(x,y) := (n+1-u_2(n+1-y, n+1-x), n+1-u_1(n+1-y, n+1-x))\ , \tag{10}$$

$${}^\pi u(x,y) := (n+1-u_1(n+1-x, n+1-y), n+1-u_2(n+1-x, n+1-y)) \tag{11}$$

for all $(x,y) \in [n]^2$. Equivalently, ${}^a u = (w_0 \otimes w_0)\,({}^t u)\,(w_0 \otimes w_0)$, and ${}^\pi u = (w_0 \otimes w_0)u(w_0 \otimes w_0)$ where w_0 denotes the longest permutation in S_n, namely $w_0 := n \cdots 321$. Notice that the diagram of ${}^t u$ (resp. ${}^a u, {}^\pi u$) is obtained by transposing (resp. reflecting through the antidiagonal, rotating by π) that of u. Moreover, it is clear that it holds $u = {}^t({}^t u) = {}^a({}^a u) = {}^\pi({}^\pi u)$, and that ${}^t(uv) = {}^t u {}^t v$, ${}^a(uv) = {}^a u {}^a v$, ${}^\pi(uv) = {}^\pi u {}^\pi v$ for all $u, v \in S([n]^2)$.

Proposition 5.13. *Let $u, v \in S([n]^2)$. Then the following conditions are equivalent.*

(1) $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$;

(2) $(1 \otimes {}^t v)({}^t u \otimes 1) = ({}^t u \otimes 1)(1 \otimes {}^t v)$;

(3) $(1 \otimes {}^\pi u)({}^\pi v \otimes 1) = ({}^\pi v \otimes 1)(1 \otimes {}^\pi u)$.

(4) $(1 \otimes {}^a v)({}^a u \otimes 1) = ({}^a u \otimes 1)(1 \otimes {}^a v)$;

Proof. We know from Proposition 4.8 that $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$ if and only if

$$\begin{array}{rcl} v_1(\alpha, \beta) & = & v_1(\alpha, u_1(\beta, \gamma)) \\ u_1(v_2(\alpha, \beta), \gamma) & = & v_2(\alpha, u_1(\beta, \gamma)) \\ u_2(v_2(\alpha, \beta), \gamma) & = & u_2(\beta, \gamma) \end{array}$$

for all $(\alpha, \beta, \gamma) \in [n]^3$. Similarly, $(1 \otimes {}^t v)({}^t u \otimes 1) = ({}^t u \otimes 1)(1 \otimes {}^t v)$ if and only if

$$\begin{array}{rcl} u_2(\beta, \alpha) & = & u_2(v_2(\gamma, \beta), \alpha) \\ v_2(\gamma, u_1(\beta, \alpha)) & = & u_1(v_2(\gamma, \beta), \alpha) \\ v_1(\gamma, u_1(\beta, \alpha)) & = & v_1(\gamma, \beta) \end{array}$$

for all $(\alpha, \beta, \gamma) \in [n]^3$. This shows the equivalence of (1) and (2). That of (1) and (3) follows immediately by taking $\sigma = n\, n-1 \cdots 321$ in Proposition 5.11, and the equivalence of (1) and (4) follows from the other two after observing that ${}^a u = {}^t({}^\pi u)$. □

Proposition 5.14. *Let $u, v \in S([n]^2)$. Then the following are equivalent:*

i) $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$;

ii) $(1 \otimes u^r)(v^r \otimes 1) = (v^r \otimes 1)(1 \otimes u^r)$*, for all* $r \in \mathbb{Z}$.

Proof. Suppose *i)* holds. Then

$$\begin{aligned} (v^{-1} \otimes 1)(1 \otimes u^{-1}) &= (v \otimes 1)^{-1}(1 \otimes u)^{-1} = \big((1 \otimes u)(v \otimes 1)\big)^{-1} \\ &= \big((v \otimes 1)(1 \otimes u)\big)^{-1} = (1 \otimes u)^{-1}(v \otimes 1)^{-1} \\ &= (1 \otimes u^{-1})(v^{-1} \otimes 1) \end{aligned}$$

and *ii)* follows since $(1 \otimes w^r) = (1 \otimes w)^r$ and $(w^r \otimes 1) = (w \otimes 1)^r$ for all $r \in \mathbb{Z}$ and $w \in S([n]^2)$. □

The following elementary result gives a simple sufficient condition for compatibility. It will be useful for the enumerative results about stable permutations.

Proposition 5.15. *Let $u, v \in S([n]^2)$ be such that $u_1(x, y) \neq v_2(a, b)$ for all $(x, y), (a, b) \in [n]^2$ with $u(x, y) \neq (x, y)$ and $v(a, b) \neq (a, b)$. Then $(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u)$.*

Proof. Let $(x,y,z) \in [n]^3$. If $v(x,y) = (x,y)$ and $u(y,z) = (y,z)$ then

$$(1 \otimes u)(v \otimes 1)(x,y,z) = (x,y,z) = (v \otimes 1)(1 \otimes u)(x,y,z) \ .$$

If $v(x,y) \neq (x,y)$ then there exists $(a,b) \in [n]^2$ such that $(a,b) \neq v(a,b) = (x,y)$. Hence $v_1(a,b) = x$ and $v_2(a,b) = y$. We therefore have that

$$(1 \otimes u)(v \otimes 1)(x,y,z) = (1 \otimes u)(v_1(x,y), v_2(x,y), z) = (v_1(x,y), v_2(x,y), z)$$

(for if $u(v_2(x,y),z) \neq (v_2(x,y),z)$ then there exists $(c,d) \in [n]^2$ such that $(c,d) \neq u(c,d) = (v_2(x,y),z)$ so $u_1(c,d) = v_2(x,y)$ which contradicts our hypotheses). Similarly, $u(y,z) = (y,z)$ (else there exists $(e,f) \in [n]^2$ such that $(e,f) \neq u(e,f) = (y,z)$ so $y = u_1(e,f)$ which is a contradiction since $y = v_2(a,b)$). Hence

$$(v \otimes 1)(1 \otimes u)(x,y,z) = (v \otimes 1)(x,y,z) = (v_1(x,y), v_2(x,y), z) \ .$$

Similarly if $u(y,z) \neq (y,z)$. $\square$

Like all results about compatibility, Proposition 5.15 has the following immediate consequence for stable permutations of rank 1.

Corollary 5.16. *Let $u \in S([n]^2)$ be such that $u_1(x,y) \neq u_2(a,b)$ for all $(x,y),(a,b) \in [n]^2$ with $u(x,y) \neq (x,y)$ and $u(a,b) \neq (a,b)$. Then u is stable of rank 1.*

Note that the class of permutations to which Corollary 5.16 applies can be equivalently described as those permutations $u \in S([n]^2)$ such that if $(a,b) \in [n]^2$ and $u(a,b) \neq (a,b)$ then $u(x,a) = (x,a)$ and $u(b,x) = (b,x)$ for all $x \in [n]$.

We conclude this section with the following result that is a refinement of Proposition 5.10 in a special case.

Proposition 5.17. *Let $i \in \{2,\ldots,n-1\}$ and $u = ((i-1,i+1),(i,i+1))((i+1,i-1),(i+1,i))$. Then u is stable of rank 1.*

Proof. We verify that $(u \otimes 1)(1 \otimes u) = (1 \otimes u)(u \otimes 1)$. Note that

$$1 \otimes u = \prod_{x=1}^{n} ((x,i-1,i+1),(x,i,i+1)) \ ((x,i+1,i-1),(x,i+1,i)).$$

Therefore (where, for notational simplicity, we write $(u(a,b),c)$ in place of

(x, y, c), if $u(a, b) = (x, y))$

$$
\begin{aligned}
&(u \otimes 1)(1 \otimes u)(u \otimes 1) \\
&= \prod_{x=1}^{n} \big((u(x, i-1), i+1), (u(x, i), i+1)\big)\ \big((u(x, i+1), i-1), (u(x, i+1), i)\big) \\
&= \prod_{x \in [n] \setminus \{i+1\}} \big((x, i-1, i+1), (x, i, i+1)\big)\ \big((u(i+1, i-1), i+1), (u(i+1, i), i+1)\big) \\
&\quad \times \prod_{x \in [n] \setminus \{i-1, i\}} \big((x, i+1, i-1), (x, i+1, i)\big) \\
&\quad \times \big((u(i-1, i+1), i-1), (u(i-1, i+1), i)\big)\ \big((u(i, i+1), i-1), (u(i, i+1), i)\big) \\
&= \prod_{x \in [n] \setminus \{i+1\}} \big((x, i-1, i+1), (x, i, i+1)\big)\ \big((i+1, i, i+1), (i+1, i-1, i+1)\big) \\
&\quad \times \prod_{x \in [n] \setminus \{i-1, i\}} \big((x, i+1, i-1), (x, i+1, i)\big) \\
&\quad \times \big((i, i+1, i-1), (i, i+1, i)\big)\ \big((i-1, i+1, i-1), (i-1, i+1, i)\big) \\
&= 1 \otimes u
\end{aligned}
$$

as claimed. □

This result could also be deduced from repeated applications of Proposition 7.5 and the case $n = 3$.

6 Symmetries

In this section we discuss various issues related to actions on stable permutations.

6.1 Symmetries of stable permutations of rank 1 (case $r = 2$)

We now consider some operations that preserve the property of being stable. From the general analysis in the previous section we can immediately draw the following consequence.

Corollary 6.1. *Let $u \in S([n]^2)$ and $\sigma \in S_n$, then the following conditions are equivalent:*

1. *u is stable of rank 1;*
2. *u^σ is stable of rank 1;*

3. ${}^{t}u$ *is stable of rank 1;*

4. ${}^{\pi}u$ *is stable of rank 1;*

5. ${}^{a}u$ *is stable of rank 1;*

6. u^r *is stable of rank 1, for all* $r \in \mathbb{Z}$.

Note that it is not true in general that if u is stable of rank 1 then the permutation ${}^{\pi/2}u$ whose diagram is the diagram of u rotated by $\pi/2$ counterclockwise is still stable. For example, if $u = ((1,4),(2,3))$, one can verify that u is stable of rank 1 but ${}^{\pi/2}u = ((1,1),(2,2))$ is not stable. Similarly, the permutation $u' = ((3,3),(4,4))$ whose diagram is obtained by reflecting the diagram of u through the middle horizontal axis is not stable. (See also Theorem 8.1 below.)

6.2 Symmetries of stable permutations

We begin with three results that are simple consequences of the general analysis made in [9]. We include short proofs for the benefit of the reader in the language of Cuntz algebras. Self-contained proofs are possible, but longer.

Proposition 6.2. *If* $u \in S([n]^r)$ *is stable then*

$$(v \otimes 1)u(1 \otimes v^{-1}) \in S([n]^r) \tag{12}$$

is stable for all $v \in S([n]^{r-1})$.

Proof. If u is stable, namely $\lambda_u \in \mathrm{Aut}(\mathcal{O}_n)$ then $\mathrm{Ad}(v) \circ \lambda_u = \lambda_{vu\varphi(v^{-1})} \in \mathrm{Aut}(\mathcal{O}_n)$, that is $(v \otimes 1)u(1 \otimes v^{-1})$ is stable. □

In particular, all the permutations that can be written in the form $(v \otimes 1)(1 \otimes v^{-1})$ for some $v \in S([n]^{r-1})$ are stable (they correspond to the inner automorphisms of $\mathcal{O}_n$), and one can compute that they have rank $\leq r$. Indeed,

$$\begin{aligned}\psi_r(v\varphi(v^{-1})) &= \varphi^r(\varphi(v)v^{-1}) \cdots \varphi(\varphi(v)v^{-1})\varphi(v)v^{-1}\varphi(v\varphi(v^{-1})) \cdots \varphi^r(v\varphi(v^{-1}))\\ &= \varphi^{r+1}(v)v^{-1}\varphi(v)\varphi^{r+1}(v^{-1})\\ &= v^{-1}\varphi(v)\\ &= \psi_{r-1}(v\varphi(v^{-1})).\end{aligned}$$

The following is a simple consequence of the last result, which we think deserves to be mentioned explicitly.

Corollary 6.3. *Let* $u \in S_n^{\otimes r}$, *then* u *is stable. In particular,* $N_n^r \geq (n!)^r$.

Proof. Let $u = u_1 \otimes u_2 \otimes \cdots \otimes u_r$, where $u_i \in S_n$ for all $i = 1, \dots, r$ then $u = (v_1 \otimes v_2 \otimes \cdots \otimes v_{r-1} \otimes 1)(v \otimes 1 \otimes \cdots \otimes 1)(1 \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_{r-1})^{-1}$, where $v, v_i \in S_n$, are given by $v = u_r \cdots u_2 u_1$ and $v_i = u_{i+1}^{-1} \cdots u_{r-1}^{-1} u_r^{-1}$, $i = 1, \dots, r-1$, so u is stable by Proposition 6.2. ☐

The following result implies that, if $r = 2$, then there is an action of $S_n \times S_n \times S_n$ on the set of stable permutations.

Lemma 6.4. *Let $u \in S([n]^r)$ be a stable permutation, and $\sigma \in S_n$. Then $u\sigma$ is a stable permutation, as well as $\varphi^{r-1}(\sigma)u$. If $u \in S([n]^2)$ is stable, then $(\sigma \otimes \tau)u(\rho \otimes \sigma^{-1})$ is stable, for all $\sigma, \tau, \rho \in S_n$.*

Proof. The first statement follows from the equalities $\lambda_u \lambda_\sigma = \lambda_{\lambda_u(\sigma)u} = \lambda_{u\sigma u^* u} = \lambda_{u\sigma}$ and $\lambda_\sigma \lambda_u = \lambda_{\lambda_\sigma(u)\sigma} = \mathrm{Ad}_{\sigma\varphi(\sigma)\dots\varphi^{r-2}(\sigma)} \lambda_{\varphi^{r-1}(\sigma)u}$. The second statement follows from the first one with $r = 2$ and Proposition 6.2, by writing $(\sigma \otimes \tau)u(\rho \otimes \sigma^{-1}) = (\sigma \otimes 1)(1 \otimes \tau)u(\rho \otimes 1)(1 \otimes \sigma^{-1})$. ☐

Note that the above action does not preserve the rank. For example, if $u \in S([n]^2)$ is stable of rank 1 and $\sigma, \tau \in S_n$, then $\sigma u \tau \varphi(\sigma)^*$ is not necessarily of rank 1. Indeed, it is not difficult to check that this happens if and only if

$$u\varphi(u)\varphi(\tau) = \varphi(\sigma u \tau) u \varphi(\sigma)^* \ .$$

For instance, for $u = 1$, then $\sigma\tau\varphi(\sigma)^*$ is of rank 1 if and only if σ and τ commute (see Prop. 4.10).

Also, the above action does not preserve compatibility. That is, if u is compatible with v and $\sigma, \tau, \rho \in S_n$ then it is not necessarily true that u is compatible with $(\sigma \otimes \tau)v(\rho \otimes \sigma^{-1})$. For example, if $n = 3$, $u = v = ((1,2),(1,3))$, $\sigma = \rho = 123$ and $\tau = (1,2)$ then one can check that u is compatible with u (see also Theorem 8.1 below) but u is not compatible with $(1 \otimes \tau)u$, since $1 \otimes u$ and $((1 \otimes \tau)u) \otimes 1$ don't commute. However, it is not difficult to check that if u is compatible with v and $\rho \in S_n$ then u is compatible with $v(\rho \otimes 1)$.

For each stable permutation $u \in S([n]^2)$, we consider the double orbit

$$\{\sigma u \tau \varphi(\sigma)^* \mid \sigma, \tau \in S_n\} \ , \tag{13}$$

consisting of $(n!)^2$ stable permutations, as shown below. For instance, the double orbit of $1 \in S([n]^2)$ is $S_n \otimes S_n$.

Proposition 6.5. *For any stable permutation $v \in S([n]^r)$, $r \geq 2$, the map $S_n \times S_n \ni (\sigma, \tau) \mapsto \sigma v \tau \varphi(\sigma)^* \in S([n]^r)$ is injective.*

Proof. Suppose $\mathrm{Ad}(\sigma)\lambda_{v\tau} = \mathrm{Ad}(\sigma')\lambda_{v\tau'}$, where $\sigma, \tau, \sigma', \tau' \in S_n$. This means that $\mathrm{Ad}(\sigma'^*\sigma)\lambda_v\lambda_\tau = \lambda_v\lambda_{\tau'}$ and therefore

$$\mathrm{Ad}\big((\lambda_v)^{-1}(\sigma'^*\sigma)\big)\lambda_\tau = (\lambda_v)^{-1}\mathrm{Ad}(\sigma'^*\sigma)\lambda_v\lambda_\tau = \lambda_{\tau'} \ .$$

Outerness of nontrivial Bogolubov automorphisms implies that $\tau = \tau'$, and we get $\mathrm{Ad}(\sigma)\lambda_v = \mathrm{Ad}(\sigma')\lambda_v$, i.e. $\mathrm{Ad}(\sigma'^*\sigma) = \mathrm{id}$. Therefore, being $\mathcal{O}_n$ simple, $\sigma'^*\sigma \in \mathbb{C}1$, and hence $\sigma = \sigma'$. □

In contrast, the cardinality of the "triple orbits",

$$\{\sigma\varphi(\rho)u\tau\varphi(\sigma)^* \mid \sigma, \rho, \tau \in S_n\} \ ,$$

($u \in S([n]^2)$) is not constant, in general.

Note that the hypothesis that v is stable is necessary in Proposition 6.5. For example, if $n = 3$, $v = ((1,2),(2,1))$, and $\sigma = (1,2)$ then $v = (\sigma \otimes 1)v(1 \otimes \sigma^{-1})$.

Finally, note that if $x, y, u, v \in S([n]^2)$, x is in the double orbit of u, and y is in the double orbit of v, then it is not necessarily true that xy is in the double orbit of uv, even if u and v are stable, and u is compatible with v. For example, if $n = 4$, $u = ((1,2),(1,3))$, $v = ((1,3),(1,4))$, $\sigma = 2134$, $x := (\sigma \otimes 1)u(1 \otimes \sigma^{-1})$, and $y := (\sigma \otimes 1)v(1 \otimes \sigma^{-1})$ then one can check (preferably with the aid of a computer) that xy is not in the double orbit of uv.

We conclude by examining, for stable permutations, the symmetries considered in the previous section.

Proposition 6.6. *Let $u \in S([n]^2)$ and $\sigma \in S_n$. Then the following are equivalent:*

i) u is stable;

ii) u^σ is stable;

iii) ${}^\pi u$ is stable.

Proof. The equivalence of i) and ii) is a simple consequence of Lemma 6.4 and Proposition 6.2. Indeed, since u is stable we have from Lemma 6.4 that $(1 \otimes v^{-1})u$ is stable, and then by Proposition 6.2 that $(v^{-1} \otimes 1)((1 \otimes v^{-1})u)(v \otimes v)$ is stable, as claimed. The equivalence of ii) and iii) follows immediately from the previous equivalence since ${}^\pi u = u^{\omega_0}$ where $\omega_0 = n \cdots 21$. □

Note that it is not true that if u is stable then its transpose ${}^t u$ is stable. For example, if $u = 123856479$ (so u is the permutation depicted at the bottom right of Figure 2) then ${}^t u = 162453789$ and one may check that this is not a stable permutation. Also, it is not true that if v is a stable permutation then v^{-1} is stable. Indeed, $({}^t u)^{-1} = 136452789$ is stable (it is the permutation depicted at the bottom left of Figure 2, and is also the rotation by π of u, which is stable by the previous proposition). Reflection of the box diagram through a vertical or horizontal axis also does not preserve stability. For example, if $n = 3$ then $((3,1),(3,2))$ is stable, while $((1,1),(1,2))$ is not and $((1,2),(1,3))$ is (see also Theorem 8.1 below).

7 Immersions

In this section we examine another combinatorial operation that preserves stability. More precisely, we show that some of the ways to naturally embed $S([n]^2)$ in $S([n+1]^2)$ preserve the property of being stable.

For $a, i \in \mathbb{N}$ let

$$a^{(i)} := a - \chi(a \geq i)$$

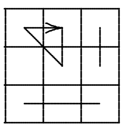
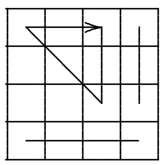
and

$$a^{<i>} := a + \chi(a \geq i) \ .$$

Note that $(a^{<i>})^{(i)} = a$ for all $a \in \mathbb{N}$, and that the function $a \mapsto a^{<i>}$ is injective. More generally, we let $(a_1, \ldots, a_k)^{<i>} := (a_1^{<i>}, \ldots, a_k^{<i>})$, for all $a_1, \ldots, a_k \in \mathbb{N}$. For $u \in S([n]^r)$ and $i \in [n+1]$ we let $u^{(i)} \in S([n+1]^r)$ be defined by

$$u^{(i)}(x_1, \ldots, x_r) := \begin{cases} (x_1, \ldots, x_r), & \text{if } x_j = i \text{ for some } j \in [r], \\ (u_1(x_1^{(i)}, \ldots, x_r^{(i)}), \ldots, u_r(x_1^{(i)}, \ldots, x_r^{(i)}))^{<i>}, & \text{otherwise} \end{cases}$$

for all $(x_1, \ldots, x_r) \in [n+1]^r$. We call $u^{(i)}$, $i = 1, \ldots, n+1$ the i-th *immersion* of u. So for example, if $u \in S([3]^2)$ is the permutation whose box diagram is depicted below, then $u^{(2)}$ is the permutation of $S([4]^2)$ whose box diagram is depicted to its right.

$u =$  $u^{(2)} =$ 

Note that

$$u^{(i)}(x^{<i>}, y^{<i>}) = u(x,y)^{<i>} \tag{14}$$

for all $x, y \in [n]$.

Remark 7.1. Note that any two different immersions $u^{(i)}$ and $u^{(j)}$ are conjugated through a permutation of the form $p \otimes p \in S([n+1]^2)$, for a suitable $p = p_{ij} \in S([n+1])$. Indeed, for $i, j \in [n+1]$ let $p := p_{i,j} \in S([n+1])$ be defined by

$$p(x) := \begin{cases} (x^{(i)})^{<j>}, & \text{if } x \neq i, \\ j, & \text{if } x = i, \end{cases}$$

for all $x \in [n+1]$. Then it is not hard to check that $(p \otimes p)u^{(i)} = u^{(j)}(p \otimes p)$ in $S([n+1]^2)$.

Remark 7.2. If U is the $n^2 \times n^2$ permutation matrix corresponding to $u \in S([n]^2)$ and we index the rows and columns of U by $[n] \times [n]$ (where this set has the lexicographic order) then the $(n+1)^2 \times (n+1)^2$ permutation matrix $U^{(i)}$ corresponding to $u^{(i)} \in S([n+1]^2)$ is obtained by putting a 1 on the main diagonal and 0's elwewhere in every row indexed by $(k,l) \in [n+1] \times [n+1]$ such that $i = k$ or $i = l$ (or both) and placing the matrix U in the remaining positions in the natural way.

We begin by noting that products and compatibility are well behaved under immersions. The next result follows immediately from our definition and we omit its verification.

Lemma 7.3. *Let $u, v \in S([n]^2)$ and $i \in [n+1]$. Then*

$$(uv)^{(i)} = u^{(i)}v^{(i)}$$

in $S([n+1]^2)$.

Proposition 7.4. *Let $u, v \in S([n]^2)$ and $i \in [n+1]$. Then*

$$(1 \otimes u^{(i)})(v^{(i)} \otimes 1) = (v^{(i)} \otimes 1)(1 \otimes u^{(i)})$$

if and only if

$$(1 \otimes u)(v \otimes 1) = (v \otimes 1)(1 \otimes u) .$$

Proof. Let $(x, y, z) \in [n+1]^3$. If $x = i$ or $y = i$ or $z = i$ then it is clear that $(1 \otimes u^{(i)})(v^{(i)} \otimes 1)(x, y, z) = (v^{(i)} \otimes 1)(1 \otimes u^{(i)})(x, y, z)$. Let $(x, y, z) \in ([n+1] \setminus \{i\})^3$. We then have that

$$\begin{aligned}(v^{(i)} \otimes 1)(1 \otimes u^{(i)})(x, y, z) &= (v^{(i)} \otimes 1)(x, u_1(y^{(i)}, z^{(i)})^{<i>}, u_2(y^{(i)}, z^{(i)})^{<i>}) \\ &= (v_1(x^{(i)}, u_1(y^{(i)}, z^{(i)}))^{<i>}, v_2(x^{(i)}, u_1(y^{(i)}, z^{(i)}))^{<i>}, u_2(y^{(i)}, z^{(i)})^{<i>}) ,\end{aligned}$$

while

$$(1\otimes u^{(i)})(v^{(i)}\otimes 1)(x,y,z) = (1\otimes u^{(i)})(v_1(x^{(i)},y^{(i)})^{<i>}, v_2(x^{(i)},y^{(i)})^{<i>}, z)$$
$$= (v_1(x^{(i)},y^{(i)})^{<i>}, u_1(v_2(x^{(i)},y^{(i)}),z^{(i)})^{<i>}, u_2(v_2(x^{(i)},y^{(i)}),z^{(i)})^{<i>}) \ .$$

Hence, $(v^{(i)}\otimes 1)(1\otimes u^{(i)}) = (1\otimes u^{(i)})(v^{(i)}\otimes 1)$ if and only if

$$v_1(x^{(i)}, u_1(y^{(i)}, z^{(i)})) = v_1(x^{(i)}, y^{(i)}) \ , \tag{15}$$

$$v_2(x^{(i)}, u_1(y^{(i)}, z^{(i)})) = u_1(v_2(x^{(i)}, y^{(i)}), z^{(i)}) \ , \tag{16}$$

$$u_2(y^{(i)}, z^{(i)}) = u_2(v_2(x^{(i)}, y^{(i)}), z^{(i)}) \ , \tag{17}$$

for all $(x,y,z) \in ([n+1]\setminus\{i\})^3$. But the function $a \mapsto a^{(i)}$ is a bijection between $[n+1]\setminus\{i\}$ and $[n]$. Therefore, equations (15), (16) and (17) hold for all $(x,y,z) \in \big([n+1]\setminus\{i\}\big)^3$ if and only if

$$v_1(a, u_1(b,c)) = v_1(a,b) \ ,$$

$$v_2(a, u_1(b,c)) = u_1(v_2(a,b),c) \ ,$$

$$u_2(b,c) = u_2(v_2(a,b),c) \ ,$$

for all $(a,b,c) \in [n]^3$. The result hence follows from Proposition 4.8. □

As an immediate corollary of Propositions 7.4 and 4.5 we obtain that the operation of immersion preserves the property of being stable of rank 1.

Corollary 7.5. *Let $u \in S([n]^2)$ and $i \in [n+1]$. Then u is stable of rank 1 if and only if $u^{(i)}$ is stable of rank 1.*

We now show that stability itself is preserved under immersions. The proof requires some preliminary lemmas. Recall the definition of $\psi_k(u)$ for $u \in S([n]^2)$ and $k \in \mathbb{N}$ from Section 4.1.

Lemma 7.6. *Let $k \geq 0$, $z_1,\ldots,z_{k+2} \in [n]$, $u \in S([n]^2)$, and $i \in [n+1]$. Then*

$$\psi_k(u^{(i)})((z_1,\ldots,z_{k+2})^{<i>}) = (y_1,\ldots,y_{k+2})^{<i>}$$

where $(y_1,\ldots,y_{k+2}) = \psi_k(u)(z_1,\ldots,z_{k+2})$.

Proof. If $k=0$ then we have from Lemma 7.3, and the fact that $1^{(i)} = 1$, that

$$\psi_0(u^{(i)}) = (u^{(i)})^{-1} = (u^{-1})^{(i)} = \psi_0(u)^{(i)}$$

and hence $\psi_0(u^{(i)})(z_1^{<i>}, z_2^{<i>}) = \psi_0(u)^{(i)}(z_1^{<i>}, z_2^{<i>}) = (\psi_0(u)(z_1, z_2))^{<i>}$, as claimed. Assume $k \geq 1$. Let, for convenience, $v := u^{-1}$, and let $(y_1, \ldots, y_{k+1}) := \psi_{k-1}(u)(z_1, \ldots, z_k, u_1(z_{k+1}, z_{k+2}))$. Then

$$\begin{aligned}
&\psi_k(u^{(i)})(z_1^{<i>}, \ldots, z_{k+2}^{<i>}) \\
&\quad = (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{(i)})^{-1})(\psi_{k-1}(u^{(i)}) \otimes 1)(z_1^{<i>}, \ldots, z_k^{<i>}, u(z_{k+1}, z_{k+2})^{<i>}) \\
&\quad = (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{(i)})^{-1})(\psi_{k-1}(u)(z_1, \ldots, z_k, u_1(z_{k+1}, z_{k+2}))^{<i>}, u_2(z_{k+1}, z_{k+2})^{<i>}) \\
&\quad = (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{-1})^{(i)})(y_1^{<i>}, \ldots, y_{k+1}^{<i>}, u_2(z_{k+1}, z_{k+2})^{<i>}) \\
&\quad = (y_1^{<i>}, \ldots, y_k^{<i>}, v_1(y_{k+1}, u_2(z_{k+1}, z_{k+2}))^{<i>}, v_2(y_{k+1}, u_2(z_{k+1}, z_{k+2}))^{<i>}) \;.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\psi_k(u)(z_1, \ldots, z_{k+2}) \\
&\quad = (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1})(\psi_{k-1}(u) \otimes 1)(z_1, \ldots, z_k, u(z_{k+1}, z_{k+2})) \\
&\quad = (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{-1})(y_1, \ldots, y_{k+1}, u_2(z_{k+1}, z_{k+2})) \\
&\quad = (y_1, \ldots, y_k, v_1(y_{k+1}, u_2(z_{k+1}, z_{k+2})), v_2(y_{k+1}, u_2(z_{k+1}, z_{k+2})))
\end{aligned}$$

and the result follows. □

The next result states that $\psi_k(u^{(i)})$ ($u \in S([n]^2)$, $i \in [n+1]$) does not change the elements weakly to the right of the rightmost element equal to i (if such an element exists).

Lemma 7.7. *Let $k \geq 0$, $1 \leq r \leq k+2$, $z_1, \ldots, z_{r-1} \in [n+1]$, $z_{r+1}, \ldots, z_{k+2} \in [n]$, $u \in S([n]^2)$, and $i \in [n+1]$. Then*

$$\psi_k(u^{(i)})(z_1, \ldots, z_{r-1}, i, z_{r+1}^{<i>}, \ldots, z_{k+2}^{<i>}) = (y_1, \ldots, y_{r-1}, i, z_{r+1}^{<i>}, \ldots, z_{k+2}^{<i>})$$

for some $y_1, \ldots, y_{r-1} \in [n+1]$.

Proof. We proceed by induction on $k \geq 0$. If $k = 0$ then $1 \leq r \leq 2$ and one has that $\psi_0(u^{(i)})(i, z_2^{<i>}) = (i, z_2^{<i>})$ and $\psi_0(u^{(i)})(z_1, i) = (z_1, i)$ for all $z_2 \in [n]$ and $z_1 \in [n+1]$. So assume $k \geq 1$. Let, for brevity $\zeta = (z_1, \ldots, z_{r-1})$. Suppose first that $r \leq k$. We then have from our induction hypothesis that

$$\begin{aligned}
&\psi_k(u^{(i)})(\zeta, i, z_{r+1}^{<i>}, \ldots, z_{k+2}^{<i>}) \\
&= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{(i)})^{-1})(\psi_{k-1}(u^{(i)}) \otimes 1)(\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes u^{(i)})(\zeta, i, z_{r+1}^{<i>}, \ldots, z_{k+2}^{<i>}) \\
&= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{-1})^{(i)})(\psi_{k-1}(u^{(i)}) \otimes 1)(\zeta, i, z_{r+1}^{<i>}, \ldots, z_{k}^{<i>}, u(z_{k+1}, z_{k+2})^{<i>}) \\
&= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{-1})^{(i)})(y_1, \ldots, y_{r-1}, i, z_{r+1}^{<i>}, \ldots, z_{k}^{<i>}, u(z_{k+1}, z_{k+2})^{<i>}) \\
&= (y_1, \ldots, y_{r-1}, i, z_{r+1}^{<i>}, \ldots, z_{k}^{<i>}, z_{k+1}^{<i>}, z_{k+2}^{<i>}) \ .
\end{aligned}$$

Suppose now that $r = k+1$. Then

$$\begin{aligned}
\psi_k(u^{(i)})(\zeta, i, z_{k+2}^{<i>}) &= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{(i)})^{-1})(\psi_{k-1}(u^{(i)}) \otimes 1)(\zeta, i, z_{k+2}^{<i>}) \\
&= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{-1})^{(i)})(y_1, \ldots, y_k, i, z_{k+2}^{<i>}) \\
&= (y_1, \ldots, y_k, i, z_{k+2}^{<i>}) \ .
\end{aligned}$$

Finally, if $r = k+2$, then

$$\begin{aligned}
\psi_k(u^{(i)})(\zeta, i) &= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{(i)})^{-1})(\psi_{k-1}(u^{(i)}) \otimes 1)(\zeta, i) \\
&= (\underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes (u^{-1})^{(i)})(y_1, \ldots, y_{k+1}, i) \\
&= (y_1, \ldots, y_{k+1}, i),
\end{aligned}$$

and the result again follows. □

We can now prove the main result of this section.

Theorem 7.8. *Let $u \in S([n]^2)$ and $i \in [n+1]$. Then u is stable if and only if $u^{(i)}$ is stable.*

Proof. Let u be stable. Let $k \in \mathbb{N}$ be such that $\psi_k(u) = v \otimes 1$ for some $v \in S([n]^{k+1})$. Let $\widetilde{v} \in S([n+1]^{k+2})$ be defined by

$$\widetilde{v}(x_1, \ldots, x_{k+2}) := \begin{cases} (x_1, \ldots, x_{k+2}), & \text{if } 1 \leq r \leq 2, \\ (\psi_{r-3}(u^{(i)})(x_1, \ldots, x_{r-1}), x_r, \ldots, x_{k+2}), & \text{if } 3 \leq r \leq k+2, \\ (v^{(i)}(x_1, \ldots, x_{k+1}), x_{k+2}), & \text{if } r = k+3, \end{cases}$$

for $(x_1,\dots,x_{k+2}) \in [n+1]^{k+2}$, where $r := \min\{j \in [k+2] : x_j = i\}$ (and $r := k+3$ if $\{j \in [k+2] : x_j = i\} = \emptyset$). Note that, indeed, $\widetilde{v} \in S([n+1]^{k+2})$. We claim that $\psi_{k+1}(u^{(i)}) = \widetilde{v} \otimes 1$. Let $x_1,\dots,x_{k+3} \in [n+1]$. Assume first that $x_1,\dots,x_{k+3} \in [n+1] \setminus \{i\}$. Let $z_1,\dots,z_{k+3} \in [n]$ be such that $x_j = (z_j)^{<i>}$ for $j = 1,\dots,k+3$. Then, by Lemma 7.6,

$$\begin{aligned}
\psi_{k+1}(u^{(i)})(x_1,\dots,x_{k+3}) &= \psi_{k+1}(u^{(i)})(z_1^{<i>},\dots,z_{k+3}^{<i>}) \\
&= \big(\psi_{k+1}(u)(z_1,\dots,z_{k+3})\big)^{<i>} \\
&= \big(\psi_{k+1}(u)(x_1^{(i)},\dots,x_{k+3}^{(i)})\big)^{<i>} \\
&= \big(v(x_1^{(i)},\dots,x_{k+1}^{(i)}),x_{k+2}^{(i)},x_{k+3}^{(i)}\big)^{<i>} \\
&= \big(v(x_1^{(i)},\dots,x_{k+1}^{(i)})^{<i>},x_{k+2},x_{k+3}\big) \\
&= \big(v^{(i)} \otimes 1 \otimes 1\big)(x_1,\dots,x_{k+3}) \\
&= \big(\tilde{v} \otimes 1\big)(x_1,\dots,x_{k+3}) \ .
\end{aligned}$$

Suppose now that $x_j = i$ for some $1 \le j \le k+3$. Let $t := \min\{j \in [k+3] : x_j = i\}$. Define $\psi_{k,s}(u)$, $1 \le s \le k$ for brevity as

$$(\underbrace{1\otimes\cdots\otimes 1}_{k}\otimes u)(\underbrace{1\otimes\cdots\otimes 1}_{k-1}\otimes u\otimes 1)\cdots(\underbrace{1\otimes\cdots\otimes 1}_{s}\otimes u\otimes\underbrace{1\otimes\cdots\otimes 1}_{k-s}).$$

Then

$$\psi_h(u) = \psi_{h,\,s}(u^{-1})\,(\psi_{s-1}(u)\otimes\underbrace{1\otimes\cdots\otimes 1}_{h-s+1})\,\psi_{h,\,s}(u^{-1})^{-1}.$$

Let $(y_{t+1},\dots,y_{k+3}) :=$

$$\big(u^{(i)}\otimes\underbrace{1\otimes\cdots\otimes 1}_{k+1-t}\big)\cdots\big(\underbrace{1\otimes\cdots\otimes 1}_{k-t}\otimes u^{(i)}\otimes 1\big)\big(\underbrace{1\otimes\cdots\otimes 1}_{k+1-t}\otimes u^{(i)}\big)(x_{t+1},\dots,x_{k+3}).$$

Then we have, if $3 \le t \le k+2$

$$\begin{aligned}
&\psi_{k+1}(u^{(i)})(x_1,\dots,x_{t-1},i,x_{t+1},\dots,x_{k+3}) = \\
&\quad= \psi_{k+1,t}((u^{(i)})^{-1})\big(\psi_{t-1}(u^{(i)})\otimes\underbrace{1\otimes\cdots\otimes 1}_{k+2-t}\big)\big(\psi_{k+1,t}((u^{(i)})^{-1})\big)^{-1}(x_1,\dots,x_{k+3}) \\
&\quad= \psi_{k+1,t}((u^{(i)})^{-1})\big(\psi_{t-1}(u^{(i)})\otimes\underbrace{1\otimes\cdots\otimes 1}_{k+2-t}\big)(x_1,\dots,x_{t-1},i,y_{t+1},\dots,y_{k+3}) \\
&\quad= \psi_{k+1,t}((u^{(i)})^{-1})(\psi_{t-3}(u^{(i)})(x_1,\dots,x_{t-1}),i,y_{t+1},\dots,y_{k+3}) \\
&\quad= (\psi_{t-3}(u^{(i)})(x_1,\dots,x_{t-1}),i,x_{t+1},\dots,x_{k+3}) \\
&\quad= \big(\tilde{v}\otimes 1\big)(x_1,\dots,x_{k+3}) \ .
\end{aligned}$$

If $1 \leq t \leq 2$ then one can easily check that

$$\psi_{k+1}(u^{(i)})(x_1, \ldots, x_{k+3}) = (x_1, \ldots, x_{k+3}) = \big(\tilde{v} \otimes 1\big)(x_1, \ldots, x_{k+3}).$$

Finally, if $t = k+3$, let $z_1, \ldots, z_{k+2} \in [n]$ be such that $x_j = (z_j)^{<i>}$ for $j \in [k+2]$. Then one can similarly check, using Lemma 7.6, that

$$\begin{aligned}
\psi_{k+1}(u^{(i)})(x_1, \ldots, x_{k+2}, i) &= (\psi_k(u^{(i)})(x_1, \ldots, x_{k+2}), i) \\
&= (\psi_k(u^{(i)})(z_1^{<i>}, \ldots, z_{k+2}^{<i>}), i) \\
&= ((\psi_k(u)(z_1, \ldots, z_{k+2}))^{<i>}, i) \\
&= (((v \otimes 1)(z_1, \ldots, z_{k+2}))^{<i>}, i) \\
&= ((v(z_1, \ldots, z_{k+1}))^{<i>}, z_{k+2}^{<i>}, i) \\
&= (v^{(i)}(x_1, \ldots, x_{k+1}), x_{k+2}, i) \\
&= \big(\tilde{v} \otimes 1\big)(x_1, \ldots, x_{k+3}) \ .
\end{aligned}$$

Conversely, let $u^{(i)}$ be stable. Let $k \in \mathbb{N}$ be such that $\psi_k(u^{(i)}) = \tilde{v} \otimes 1$ for some $\tilde{v} \in S([n+1]^{k+1})$. Let $z_1, \ldots, z_{k+2} \in [n]$. Then, using Lemma 7.6,

$$(\psi_k(u)(z_1, \ldots, z_{k+2}))^{<i>} = \psi_k(u^{(i)})(z_1^{<i>}, \ldots, z_{k+2}^{<i>}) = (\tilde{v}(z_1^{<i>}, \ldots, z_{k+1}^{<i>}), z_{k+2}^{<i>}) \ .$$

It follows that $\tilde{v}$ restricts to a bijection of $([n+1] \setminus \{i\})^{k+1}$ and that

$$\psi_k(u)(z_1, \ldots, z_{k+2}) = (\tilde{v}(z_1^{<i>}, \ldots, z_{k+1}^{<i>})^{(i)}, z_{k+2}) \ .$$

But, because $\tilde{v}$ restricts to a bijection of $([n+1] \setminus \{i\})^{k+1}$, $(z_1, \ldots, z_{k+1}) \mapsto \tilde{v}(z_1^{<i>}, \ldots, z_{k+1}^{<i>})^{(i)}$ is a bijection v of $[n]^{k+1}$ such that $\psi_k(u) = v \otimes 1$. The result then follows from Proposition 4.4. $\square$

It seems plausible that an analogous statement works in general for $u \in S([n]^r)$. We leave this for a future investigation

8 Explicit Characterizations

In this section we obtain explicit characterizations of various classes of stable permutations. More precisely, we characterize the transpositions of $S([n]^2)$ that are stable, the 2 and 3-cycles of $S([n]^2)$ that are stable of rank 1, and give a sufficient condition for a cycle to be stable of rank 1.

8.1 Transpositions

Probably the first class of permutations that comes to mind is that of transpositions. In this subsection, we completely characterize the stable transpositions. In particular, we show that all stable transpositions have rank 1.

Theorem 8.1. *Let $(i,j),(a,b) \in [n]^2$, $(i,j) \neq (a,b)$, and $u := ((i,j),(a,b))$. Then the following conditions are equivalent:*

i) u is stable;

ii) u is stable of rank 1;

iii) $\{a,i\} \cap \{b,j\} = \emptyset$.

Proof. We show first that *iii)* implies *ii)*. Assume that $\{a,i\} \cap \{b,j\} = \emptyset$.

Then we have that

$$\begin{aligned}(1 \otimes u)(u \otimes 1)(1 \otimes u^{-1}) &= (1 \otimes u) \prod_{x=1}^{n} ((a,b,x),(i,j,x))(1 \otimes u^{-1}) \\ &= \prod_{x=1}^{n} ((a,u(b,x)),(i,u(j,x))) \\ &= \prod_{x=1}^{n} ((a,b,x),(i,j,x)) \\ &= (u \otimes 1)\end{aligned}$$

where we have used the fact that $u(b,x) = (b,x)$ and $u(j,x) = (j,x)$ for all $x \in [n]$ since $\{a,i\} \cap \{b,j\} = \emptyset$. So, by Proposition 4.5, u is stable of rank 1.

It is clear that *ii)* implies *i)*. We now show that *i)* implies *iii)*. Assume that $\{a,i\} \cap \{b,j\} \neq \emptyset$. We may clearly assume that $(a,b) < (i,j)$ in lexicographic order. We have six cases to distinguish.

i) $a = i$

Then $b < j$. If $a = i = b$ then, for any $k \in \mathbb{N}$,

$$\psi_k(u)(\underbrace{a,a,\ldots,a}_{k+2}) = (a,j,\ldots,j)$$

so u is not stable. Similarly if $a = i = j$.

ii) $b = j$

Then $a < i$. If $b = j = a$ then, for any $r \in \mathbb{N}$,

$$\psi_{2r-1}(u)(\underbrace{a,a,\ldots,a}_{2r},i) = (i,\underbrace{a,a,\ldots,a}_{2r-1},i),$$

while

$$\psi_{2r-1}(u)(\underbrace{a, a, \ldots, a}_{2r}, a) = (a, a, \ldots, a, a),$$

so $\psi_{2r-1}(u) \notin S([n]^{2r}) \otimes 1$ for all $r \in \mathbb{N}$ and hence, by Proposition 4.4, u is not stable. Similarly if $b = j = i$.

iii) $a \neq i$, $b \neq j$, $a = b$

Then $u(j, a) = (j, a)$ so, for any $r \in \mathbb{N}$,

$$\psi_{2r-1}(u)(\underbrace{i, j, \ldots, i, j}_{2r}, a) = (a, \underbrace{i, j, \ldots, i, j}_{2r}),$$

so, as in the previous case, u is not stable.

iv) $a \neq i$, $b \neq j$, $a = j$

Then $a \neq b$, so $u(b, b) = (b, b)$ and hence, for any $k \in \mathbb{N}$,

$$\psi_k(u)(\underbrace{i, i, \ldots, i}_{k+1}, a) = (a, \underbrace{b, b, \ldots, b}_{k+1}),$$

so u is not stable.

v) $a \neq i$, $b \neq j$, $i = b$

Then we conclude as in case *iv)* above.

vi) $a \neq i$, $b \neq j$, $i = j$

Then we conclude as in case *iii)* above. □

Remark 8.2. Note that no transposition in $S([n]^2)$ belongs to $S_n \otimes S_n$, as every nontrivial element in $S_n \otimes S_n$ moves at least $2n$ points, which is always larger than 2. Thus, the automorphisms corresponding to the stable transpositions are all outer automorphisms of $\mathcal{O}_n$, since, as mentioned in the Introduction, the inner automorphisms correspond to the elements of the form $u \otimes u^{-1}$, $u \in S_n$.

The previous result enables us to obtain a simple closed formula for the number of stable transpositions in $S([n]^2)$.

Corollary 8.3. *In $S([n]^2)$ there are $n(n-1)^2(n-2)/2$ stable transpositions (all of rank 1).*

Proof. Since for any such transposition $((a, b), (i, j))$ one has, by Theorem 8.1, that $\{a, i\} \cap \{b, j\} = \emptyset$, it is clear that $i \neq j$ and $a \neq b$. Thus there are $n(n-1)$ possible pairs (i, j). For any fixed such pair (i, j), (a, b) must lie in a row different from the j-th, in a column different from the i-th, and not on the diagonal. There are thus $n^2 - (3n - 3) - 1$ possibilities for (a, b) (since (i, i), (j, j), and (j, i) are all distinct), and the result follows. □

Note that Corollary 8.3 implies that if we pick a transposition at random in $S([n]^2)$, then, as n goes to infinity, we obtain a stable transposition with probability 1.

We now show that the stable transpositions are all in different double orbits (recall our definition (13) of the double orbit of a stable permutation u).

Corollary 8.4. *Let $(a,b),(i,j) \in [n]^2$, $(a,b) \neq (i,j)$ and $u,v \in S_n$, $n > 2$. Then $(u \otimes 1)((a,b),(i,j))(v \otimes u^{-1})$ is not a transposition if $(u,v) \neq (e,e)$.*

Proof. Let, by contradiction, $(\alpha,\beta),(\gamma,\delta) \in [n]^2$, $(\alpha,\beta) \neq (\gamma,\delta)$ be such that $(u \otimes 1)((a,b),(i,j))(v \otimes u^{-1}) = ((\alpha,\beta),(\gamma,\delta))$. Then $(u \otimes 1)((a,b),(i,j))(u^{-1} \otimes 1)(uv \otimes u^{-1}) = ((\alpha,\beta),(\gamma,\delta))$ so we conclude that

$$((u(a),b),(u(i),j))((\alpha,\beta),(\gamma,\delta)) = (uv \otimes u^{-1}) \ . \tag{18}$$

Let $(x,y) \in [n]^2$. Note that $(uv \otimes u^{-1})(x,y) = (x,y)$ if and only if $x = u(v(x))$ and $y = u(y)$. If $u \neq e$, then u has at most $n-2$ fixed points, so $(uv \otimes u^{-1})$ has at most $n^2 - 2n$ fixed points. If $u = e$ then $v \neq e$ hence v has at most $n-2$ fixed points, so $(v \otimes 1)$ again has at most $n^2 - 2n$ fixed points. On the other hand, $((u(a),b),(u(i),j))((\alpha,\beta),(\gamma,\delta))$ has at least $n^2 - 4$ fixed points, and this contradicts equation (18) if $n > 2$. $\square$

It follows immediately from Corollaries 8.3 and 8.4, Proposition 6.5 and Lemma 6.4, that the number of stable permutations in $S([n]^2)$ is bounded from below by $(n!)^2(n(n-1)^2(n-2)/2+1)$, taking into account also the double orbit of (any one of) the Bogolubov automorphisms (i.e., the permutations of $S_n \otimes S_n$). This gives a lower bound of 252 stable permutations for $n = 3$ and 21312 for $n = 4$.

8.2 Cycles

In this subsection we examine the stability of cycles. More precisely, we point out that a natural generalization of the necessary and sufficient condition obtained in Theorem 8.1 for the stability of rank 1 of a transposition is sufficient also for a longer cycle, and prove a partial converse of this result.

The next result follows immediately from Proposition 5.15, but we feel that it should be stated explicitly.

Proposition 8.5. *Let $(a_1,b_1),\ldots,(a_r,b_r) \in [n]^2$, all distinct, be such that*

$$\{a_1,\ldots,a_r\} \cap \{b_1,\ldots,b_r\} = \emptyset.$$

Then $\big((a_1,b_1),\ldots,(a_r,b_r)\big)$ is stable of rank 1.

The following result provides a partial converse to Proposition 8.5.

Proposition 8.6. *Let $u := ((a_1, b_1), (a_2, b_2), \ldots, (a_r, b_r)) \in S([n]^2)$ be stable of rank 1, where $a_1, \ldots, a_r, b_1, \ldots, b_r \in [n]$ and $a_i \neq a_j$ if $i \neq j$ ($1 \leq i, j \leq r$). Then $\{a_1, \ldots, a_r\} \cap \{b_1, \ldots, b_r\} = \emptyset$.*

Proof. Assume, by contradiction, that $\{a_1, \ldots, a_r\} \cap \{b_1, \ldots, b_r\} \neq \emptyset$. Let $i, j \in [r]$ be such that $a_i = b_j$. Then

$$(1 \otimes u)(u \otimes 1)(a_j, a_i, b_i) = (1 \otimes u)(a_{j+1}, b_{j+1}, b_i)$$

while

$$(u \otimes 1)(1 \otimes u)(a_j, a_i, b_i) = (u \otimes 1)(a_j, a_{i+1}, b_{i+1}) = (a_j, a_{i+1}, b_{i+1})$$

(for if $u(a_j, a_{i+1}) \neq (a_j, a_{i+1})$ then, since $a_1, \ldots, a_r$ are all distinct, $b_j = a_{i+1}$ so $a_i = a_{i+1}$ which is a contradiction). But $a_j \neq a_{j+1}$ so $(1 \otimes u)(u \otimes 1) \neq (u \otimes 1)(1 \otimes u)$. □

Note that, by Proposition 6.1, Proposition 8.6 remains valid if we substitute the hypothesis $a_i \neq a_j$ if $i \neq j$ with the one $b_i \neq b_j$ if $i \neq j$.

8.3 3-cycles

In this subsection we study the stability of 3-cycles. More precisely, we characterize the 3-cycles which are stable of rank 1, and show that stability does not imply stability of rank 1 for 3-cycles.

Theorem 8.7. *Let $(a_1, b_1), (a_2, b_2), (a_3, b_3) \in [n]^2$ be distinct, then the 3-cycle $u := \big((a_1, b_1), (a_2, b_2), (a_3, b_3)\big) \in S([n]^2)$ is stable of rank 1 if and only if $\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\} = \emptyset$.*

Proof. We already know, from Proposition 8.5, that if $\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\} = \emptyset$ then u is stable of rank 1. Conversely, suppose that u is stable of rank 1. If $|\{a_1, a_2, a_3\}| = 3$ or $|\{b_1, b_2, b_3\}| = 3$ then the result follows from Proposition 8.6 and the comments following it. We may therefore assume that $\max\{|\{a_1, a_2, a_3\}|, |\{b_1, b_2, b_3\}|\} \leq 2$. Hence, it is enough to consider the case where $|\{a_1, a_2, a_3\}| = |\{b_1, b_2, b_3\}| = 2$ (indeed, if $|\{a_1, a_2, a_3\}| = 1$, then $u = \big((a_1, b_1), (a_1, b_2), (a_1, b_3)\big)$ so $|\{b_1, b_2, b_3\}| = 3$ and similarly for $|\{b_1, b_2, b_3\}| = 1$). We may henceforth assume that $a_1 \neq a_2 = a_3$ and therefore $b_2 \neq b_3$. So $\{a_1, a_2, a_3\} = \{a_1, a_2\}$,

$\{b_1, b_2, b_3\} = \{b_2, b_3\}$.
There are two main cases, namely either $b_1 = b_2$ or $b_1 = b_3$. Note that

$$(u \otimes 1)(1 \otimes u)(u^{-1} \otimes 1) = (u \otimes 1) \prod_{x=1}^{n} \big((x, a_1, b_1), (x, a_2, b_2), (x, a_3, b_3)\big)(u^{-1} \otimes 1)$$
$$= \prod_{x=1}^{n} \big((u(x, a_1), b_1), (u(x, a_2), b_2), (u(x, a_3), b_3)\big)$$

so

$$\prod_{x=1}^{n} \big((u(x, a_1), b_1), (u(x, a_2), b_2), (u(x, a_3), b_3)\big) = \prod_{y=1}^{n} \big((y, a_1, b_1), (y, a_2, b_2), (y, a_3, b_3)\big) .$$

Since the two cases are similar, we only treat one of them. Assume $b_1 = b_2$.

Let $x \in [n]$. Then there is $y \in [n]$ (depending on x) such that either $(u \otimes 1)(x, a_1, b_1) = (y, a_1, b_1)$ or $(u \otimes 1)(x, a_1, b_1) = (y, a_2, b_2)$. But, in the latter case, since $(1 \otimes u)$ commutes with $(u \otimes 1)$, $(u \otimes 1)(x, a_2, b_2) = (y, a_3, b_3)$ so $b_2 = b_3$, a contradiction. Therefore, u leaves the a_1-th column of $[n]^2$ globally invariant. Hence, $a_1 \neq b_2, b_3$ since the corresponding columns are not globally invariant.

Similarly, for any $x \in [n]$ there is $y \in [n]$ such that either $(u \otimes 1)(x, a_2, b_2) = (y, a_2, b_2)$ or $(u \otimes 1)(x, a_2, b_2) = (y, a_1, b_1)$. Again, in the latter case, $(u \otimes 1)(x, a_3, b_3) = (y, a_2, b_2)$ so $b_2 = b_3$, a contradiction. Therefore, u leaves the a_2-th column of $[n]^2$ globally invariant. Hence, $a_2 \neq b_2, b_3$. So $\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\} = \emptyset$.

□

Using the previous theorem we can easily enumerate the 3-cycles in $S([n]^2)$ which are stable of rank 1.

Corollary 8.8. *In $S([n]^2)$ there are exactly $(n)_4(n^2 - 3n + 4)/3$ 3-cycles that are stable of rank 1.*

Proof. Let $u = ((a_1, b_1), (a_2, b_2), (a_3, b_3))$ where $\{a_1, a_2, a_3\} \cap \{b_1, b_2, b_3\} = \emptyset$ ($a_1, a_2, a_3, b_1, b_2, b_3 \in [n]$), and $(a_1, b_1), (a_2, b_2), (a_3, b_3)$ are distinct. If $|\{a_1, a_2, a_3\}| = 1$ then there are n choices for this common value and b_1, b_2, b_3 must be distinct and different from a_1 so there are $(n-1)_3$ choices for these values. However, each u is thus obtained 3 times, so the number of such 3-cycles is $(n)_4/3$. If $|\{a_1, a_2, a_3\}| = 2$ then we may assume that $a_1 = a_2 \neq a_3$. So there are $(n)_2$ choices for these two values. Furthermore, there are $(n-2)_2$ possibilities for b_1 and b_2 (since we must have $b_2 \neq b_1$), and $n - 2$ for b_3. So we obtain $(n)_4(n-2)$ such 3-cycles. Finally, if $|\{a_1, a_2, a_3\}| = 3$ then these values can be chosen in

$(n)_3$ ways and correspondingly there are $(n-3)^3$ possibilities for the values of b_1, b_2, and b_3. Each such u is obtained three times in this way so the number of such 3-cycles is $(n)_4(n-3)^2/3$. □

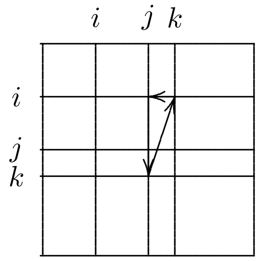
So, for example, there are 64 3-cycles that are stable of rank 1 in $S([4]^2)$. The above result easily implies that a random 3-cycle in $S([n]^2)$ is stable with probability 1, as n goes to infinity.

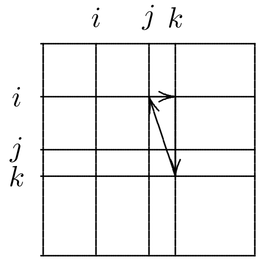
Note that a stable 3-cycle is not necessarily of rank 1. We illustrate this by considering "right-angled" 3-cycles leaving the diagonal fixed (see, for example, the permutation depicted on the left hand side of the figure below).

Proposition 8.9. *Let $i, j, k \in [n]$, i, j, k distinct. Then $((i, j), (k, j), (i, k))$ is stable of rank 2.*

Proof. Let $w := ((i, j), (k, j), (i, k))$. Then $w = ((i, k), (i, j))((i, j), (k, j))$. Furthermore $((i, k), (i, j))$ and $((i, j), (k, j))$ are stable of rank 1 by Theorem 8.1, and they are compatible by Proposition 5.6 (where $u := ((i, k), (i, j))$ and $v := ((i, j), (k, j))$, so by Theorem 5.2 u is stable of rank ≤ 2. However, by Theorem 8.7, u is not stable of rank 1. □

We conclude by noting that, on the other hand, if $i \neq j$, the 3-cycles $u := ((i, j), (i, k), (k, k))$ ($(i, j), (i, k), (k, k)$ distinct, so $k \notin \{i, j\}$) are never stable (see for example the permutation depicted on the right hand side of the figure below). Indeed, $u = ((i, j), (i, k))((i, k), (k, k))$ and, by Theorem 8.1, $((i, j), (i, k))$ is stable while $((i, k), (k, k))$ is not. On the other hand, by Proposition 5.6, $((i, j), (i, k))$ is compatible with $((i, k), (k, k))$ so, by Proposition 5.3 $((i, j), (i, k))((i, k), (k, k))$ is not stable.





9 A construction for composite n

In this section we show that the arithmetic structure of n affects the set of stable permutations. More precisely, we prove that if n is a composite number then there are stable permutations that have no counterpart when n is prime.

Let $n = n_1 n_2$. Without loss of generality, we assume that $n_1 \geq n_2$. We identify $[n_1] \times [n_2]$ with $[n]$ by identifying (a, b) with $n_2(a-1)+b$. Let $\bar{\cdot}$ be an involution of $[n_1]$ which leaves $[n_2] \subseteq [n_1]$ globally invariant. We define $u \in S([n]^2)$ by

$$u((a,b),(c,d)) := \begin{cases} ((\overline{a},b),(c,\overline{d})) & \text{if } a = d \\ ((a,b),(c,d)) & \text{if } a \neq d \end{cases}$$

for all $(a,b),(c,d) \in [n_1] \times [n_2]$. Note that u is an involution.

Proposition 9.1. *With the above notation u is stable of rank 1.*

Proof. Consider $(a,b),(c,d),(e,f) \in [n_1] \times [n_2]$. Suppose first that $a = d$ and $c = f$. Then we have that

$$\begin{aligned} \psi_1(u)((a,b),(c,d),(e,f)) &= (1 \otimes u)(u \otimes 1)((a,b),(\overline{c},d),(e,\overline{f})) = \\ &= (1 \otimes u)((\overline{a},b),(\overline{c},\overline{d}),(e,\overline{f})) = ((\overline{a},b),(c,\overline{d}),(e,f)). \end{aligned}$$

If $a = d$ and $c \neq f$ then

$$\begin{aligned} \psi_1(u)((a,b),(c,d),(e,f)) &= (1 \otimes u)(u \otimes 1)((a,b),(c,d),(e,f)) = \\ &= (1 \otimes u)((\overline{a},b),(c,\overline{d}),(e,f)) = ((\overline{a},b),(c,\overline{d}),(e,f)). \end{aligned}$$

Similarly, if $a \neq d$ and $c = f$ then

$$\begin{aligned} \psi_1(u)((a,b),(c,d),(e,f)) &= (1 \otimes u)(u \otimes 1)((a,b),(\overline{c},d),(e,\overline{f})) = \\ &= (1 \otimes u)((a,b),(\overline{c},d),(e,\overline{f})) = ((a,b),(c,d),(e,f)). \end{aligned}$$

Finally, if $a \neq d$ and $c \neq f$ then clearly

$$\psi_1(u)((a,b),(c,d),(e,f)) = ((a,b),(c,d),(e,f)) \; .$$

This shows that $\psi_1(u) = u \otimes 1$. □

Likewise, defining $v \in S([n]^2)$ by

$$v((a,b),(c,d)) := \begin{cases} ((a,\overline{b}),(\overline{c},d)) & \text{if } b = c \\ ((a,b),(c,d)) & \text{if } b \neq c \end{cases}$$

for all $(a,b),(c,d) \in [n_1] \times [n_2]$, one obtains again a stable involution. The proof is similar and is omitted.

Proposition 9.2. *With the above notation v is stable of rank 1.*

Analogous statements work for $w, z \in S([n]^2)$ defined respectively by

$$w((a,b),(c,d)) := \begin{cases} ((a,\bar{b}),(\bar{c},d)) & \text{if } b \neq c \\ ((a,b),(c,d)) & \text{if } b = c \end{cases}$$

and

$$z((a,b),(c,d)) := \begin{cases} ((\bar{a},b),(c,\bar{d})) & \text{if } a \neq d \\ ((a,b),(c,d)) & \text{if } a = d \end{cases}$$

for all $(a,b),(c,d) \in [n_1] \times [n_2]$. The details of the proofs are left to the reader.

In the case where $n = 4$, it turns out that the last two stable involutions are in the same double orbits as the first two. Also, if $n_1 = n_2$ then $v = (F \otimes F)u(F \otimes F)$ where $F : S([n_1] \times [n_1]) \to S([n_1] \times [n_1])$ is the "flip" map ($F(a,b) := (b,a)$ for all $(a,b) \in S([n_1] \times [n_1])$) so u and v are in the same triple orbit if $n_1 = n_2$. On the other hand, one can check that the triple orbit of $u \in S([4]^2)$ does not contain any permutation that fixes the diagonal of $[4]^2$ pointwise, hence u is not in the same triple orbit as any compatible product of stable transpositions.

10 Small cases

In this section we discuss, in light of the results obtained so far, the stable permutations of $S([3]^2)$ and $S([4]^2)$.

10.1 $n = 3$

One can check that for $n = 3$ there are precisely $576 = 96 \cdot 3!$ stable permutations in $S([3]^2)$ [9, 8]. Among them, there are the $36 = (3!)^2$ inner perturbations of the Bogolobov automorphisms in $S([3]) \otimes S([3])$.

It turns out that in this case the search of the permutations studied in Propositions 4.5 and 4.11 can be fully accomplished.

Proposition 10.1. *In $S([3]^2)$ there are 54 stable permutations of rank 1 and 52 permutations satisfying equation (4). Moreover, in $S([3]^2) \setminus S([3]) \otimes S([3])$, there are 36 solutions of equation (3), 36 solutions of equation (4), and these two sets coincide. These are precisely the involutions in $S([3]^2) \setminus S([3]) \otimes S([3])$ satisfying either equation.*

In detail, using Proposition 4.10, it is not difficult to see that the 18 = 54-36 permutations in $S_3 \otimes S_3$ that are stable of rank 1 consist of:
- the six permutations of the form $\{u \otimes 1 \ : \ u \in S_3\}$
- the five permutations of the form $\{1 \otimes u \ : \ u \in S_3 \setminus \{1\}\}$

- the five permutations of the form $\{u \otimes u \ : \ u \in S_3 \setminus \{1\}\}$
- the two permutations $(2,1,3) \otimes (3,1,2)$ and $(3,1,2) \otimes (2,1,3)$.

Moreover, denoting by I_n the subset of involutions in S_n, it is not difficult to see, using Proposition 4.12, that the $16 =$ 52-36 permutations in $S_3 \otimes S_3$ satisfying the equation (4) consist of:
- the four permutations of the form $\{x \otimes 1 \ : \ x \in I_3\}$
- the three permutations of the form $\{1 \otimes x \ : \ x \in I_3 \setminus \{1\}\}$
- the three permutations of the form $\{u \otimes u \ : \ u \in I_3 \setminus \{1\}\}$
- the three permutations of the form $\{x \otimes (2,1,3) \ : \ x \in I_3 \setminus \{1\}\}$
- the three permutations of the form $\{x \otimes (3,1,2) \ : \ x \in I_3 \setminus \{1\}\}$.

Thus, there are 10 permutations that are in $S_3 \otimes S_3$ and satisfy both equations (3) and (4).

Concerning the 36 involutions in $S([3]^2) \setminus S([3]) \otimes S([3])$ mentioned above, although it is not obvious how to characterize them by intrinsic structural properties (see however Proposition 5.17), it is clear that they possess some specific symmetries. For instance, by Corollary 6.1 and Proposition 3.4, the set of such involutions is invariant under the transpose map. Moreover, and we have no general explanation for this fact, they are grouped into 18 distinct pairs through the map (12), with $r = 2$, where v is any of the three transpositions of S_3, each one connecting 6 pairs. Furthermore, the transpose map respects these pairs, leaving 6 of them fixed (as a pair). In accordance with Theorem 8.1, six of these 36 permutations are transpositions, and by Corollary 8.4 these transpositions all belong to different pairs. Twelve of these 36 permutations are products of two disjoint transpositions, nine of these twelve are the immersions (see Proposition 7.4) of the stable permutations (of rank 1) of $S([2]^2) \setminus \{1\}$ and the other three are obtained from these immersions through the map (12); altogether they give rise to nine distinct pairs, different from the 6 corresponding to the transpositions above. Finally, six of these permutations are products of an immersion of the product of two parallel transpositions of $S([2]^2)$ with a non-parallel disjoint transposition having one line (i.e., a row or a column) of fixed points, giving rise to three distinct pairs, that are left invariant by transposition (see Section 13).

Recall (see Lemma 6.4, and Proposition 6.5) that for any stable permutation $u \in S([n]^2)$ its double orbit (13) consists of $(n!)^2$ stable permutations. Since $576 = (3!)^2 16$, by grouping together the 576 stable permutations in $S([3]^2)$ into double orbits, we only need to describe 16 representatives. We can pick the identity, and nine more representatives can be found among the 36 involutions considered before. Now, one can check that among the 576 stable permutations there are precisely 27 stable permutations of order 3, and that 6 of them are 3-cycles. These can be chosen as the remaining 6 representatives.

Theorem 10.2. *Let u be a stable permutation in $S([3]^2)$. Then u is in the double orbit of one of the following 16 elements: either an immersion of a stable permutation in $S([2]^2)$ (ten elements) or one of the six 3-cycles* $C_1 = ((1,2),(3,2),(1,3))$, $C_2 = ((1,2),(3,1),(3,2))$, $C_3 = ((1,2),(1,3),(2,3))$, $C_4 = ((2,1),(3,2),(3,1))$, $C_5 = ((2,1),(2,3),(1,3))$, $C_6 = ((2,1),(3,1),(2,3))$.

Equivalently, u is in the double orbit of a product of stable transpositions. More precisely, u is in the double orbit of one of the following 16 elements: either the identity, or a stable transposition (six elements), or the product of a horizontal stable transposition and a vertical stable transposition (in this order, nine elements).

The first description is quite explicit (see also Section 13). Furthermore, it is worth stressing that the C_i's can be easily expressed as products of two stable transpositions of rank 1, namely

$$\begin{aligned}
C_1 &= ((1,2),(1,3))((1,2),(3,2))\\
C_2 &= ((3,1),(3,2))((1,2),(3,2))\\
C_3 &= ((1,2),(1,3))((1,3),(2,3))\\
C_4 &= ((3,1),(3,2))((3,1),(2,1))\\
C_5 &= ((2,1),(2,3))((2,3),(1,3))\\
C_6 &= ((2,1),(2,3))((2,1),(3,1))\ .
\end{aligned}$$

Actually these are the only products between a horizontal and a vertical stable transpositions of rank 1 with one endpoint in common. In fact, such permutations are always stable by Proposition 5.6 and Theorem 5.2.

The second description is probably the most concise and elegant one. It is also the one that could conceivably be most easily generalized. Note that, as a consequence of the second description and of Proposition 5.6, every stable permutation in $S([3]^2)$ is in the double orbit of a compatible product of stable transpositions.

We conclude this subsection by observing that the subgroup generated by the stable transpositions in $S([3]^2)$ coincides with the subgroup of all elements of $S([3]^2)$ that leave the diagonal of $[3]^2$ pointwise invariant.

10.2 $n = 4$

A conceptual description of all the stable permutations in $S([4]^2)$ is much more demanding and to a certain extent not known. In this subsection we describe explicitly the stable permutations in $S([4]^2)$ that can be obtained with the results presented in this work.

With the help of a computer one can check that by immersing each of the 576 stable permutations of $S([3]^2)$ in $S([4]^2)$, each one in one of the four possible ways, 2283 different permutations (stable by Theorem 7.8) are obtained and, of these, 2211 are in different double orbits. Of the 36 stable transpositions in $S([4]^2)$, 24 are immersions of (necessarily stable) transpositions in $S([3]^2)$. The remaining 12 stable transpositions are in 12 distinct double orbits, different from the ones of the 2283 permutations above. Thus, considering the immersions of the stable permutations of $S([3]^2)$ and the stable transpositions of $S([4]^2)$, we have 2223 stable permutations lying in different double orbits.

On the other hand, by Corollary 5.4, products of stable transpositions that satisfy its hyphoteses are stable. Computing products of this kind yields 9828 stable permutations (among them, 270 are of rank 1) and they are all (compatible) products of at most eight stable transpositions. We have also checked by computer that they are all in different double orbits, and that each one of the 2283 permutations obtained above is in the same double orbit as one of these 9828. These 9828 permutations are naturally partitioned by the minimum number of stable transpositions needed to express them as a compatible product. Furthermore, 61 of these 9828 permutations are immersions of stable permutations of $S([3]^2)$, and all 155 permutations satisfying the hypotheses of Proposition 5.16 are among these 9828.

We conclude by noting that not all stable permutations of $S([4]^2)$ are in the same triple orbit as compatible products of stable transpositions. For example, the construction given in Section 9 yields that

$$((1,1),(3,2))((2,1),(4,2))((1,3),(3,4))((2,3),(4,4))$$

is stable of rank 1 (actually, it is in the same double orbit as

$$((1,2),(3,1),(2,1))((1,4),(3,3),(2,3))((2,2),(3,2),(4,1))((2,4),(3,4),(4,3))$$

also of rank 1), but is not in the triple orbit of any product of stable transpositions. Indeed (as noted at the end of Section 9) one can check that no permutation in its triple orbit fixes the diagonal of $[4]^2$, while all products of stable transpositions have this property. Again, the subgroup generated by the stable transpositions is the subgroup of $S([4]^2)$ consisting of those permutations that fix the diagonal of $[4]^2$ (it is easy to see that this holds for all n). Actually, the construction given in Section 9, together with Lemma 6.4, yields 24 stable permutations whose double orbits are distinct among themselves and different from the 9828 double orbits obtained above.

11 Enumerative Results

The enumeration of the stable permutations of $S([n]^2)$ (or, more generally, of $S([n]^r)$) is a very difficult problem. Both a recursion or some information on the generating function of these numbers would be interesting as this would shed further light on their structure. We have obtained some enumerative results in Corollaries 8.3 and 8.8 for special classes of stable permutations. In this section we show that the enumeration of stable permutations can be reduced to that of a certain subclass, and we obtain upper and lower bounds for their number.

11.1 Reducible and irreducible stable permutations

Theorem 7.8 shows that if $u \in S([n]^2)$ is such that there is an $i \in [n]$ so that $u(x,i) = (x,i)$ and $u(i,x) = (i,x)$ for all $x \in [n]$, so that $u = v^{(i)}$ for some $v \in S([n-1]^2)$, then u is stable if and only if v is stable. This motivates us to define a permutation $u \in S([n]^2)$ to be *irreducible* if there is no $i \in [n]$ so that u fixes pointwise both the i-th row and the i-th column of $[n]^2$ (i.e., so that $u(x,i) = (x,i)$ and $u(i,x) = (i,x)$ for all $x \in [n]$) and *reducible* otherwise.

Let $\mathcal{SI}(n,2)$ be the set of stable irreducible permutations of $S([n]^2)$, $\mathcal{SR}(n,2)$ the one of stable reducible permutations, and $SI(n,2) := |\mathcal{SI}(n,2)|$, $SR(n,2) := |\mathcal{SR}(n,2)|$. So, for example, $SI(1,2) = 0$, $SR(1,2) = 1$, $SI(2,2) = 3$, and $SR(2,2) = 1$. Then, clearly, $N_n^2 = SI(n,2) + SR(n,2)$. However, there is also another relation.

Proposition 11.1. *Let $n \in \mathbb{N}$, $n \geq 2$. Then*

$$SR(n,2) = 1 + \sum_{k=1}^{n-1} \binom{n}{k} SI(k,2).$$

Proof. We construct a bijection between $\mathcal{SR}(n,2) \setminus \{1\}$ and $\bigcup_{k=1}^{n-2} \binom{[n]}{k} \times \mathcal{SI}(n-k,2)$ as follows. Let $(S,\sigma) \in \bigcup_{k=1}^{n-2} \binom{[n]}{k} \times \mathcal{SI}(n-k,2)$. Let $h := |S|$ and $\{i_1,\ldots,i_h\}_< := S$ and define $\varphi(S,\sigma) := (\cdots((\sigma^{(i_1)})^{(i_2)})\cdots)^{(i_h)}$ (so, $\varphi(S,\sigma)$ is the iterated embedding of σ in positions i_1, i_2, ..., i_h). Then $\varphi(S,\sigma)$ is reducible, $\varphi(S,\sigma) \neq 1$, and, by Theorem 7.8, any reducible stable permutation of $S([n]^2) \setminus \{1\}$ is of this form. Furthermore, note that, since σ is irreducible, $i \in S$ if and only if $\varphi(S,\sigma)$ fixes pointwise both the i-th column and the i-th row of $[n]^2$. Hence if $\varphi(S,\sigma) = \varphi(T,\tau)$ for some $(S,\sigma),(T,\tau) \in \bigcup_{k=1}^{n-2} \binom{[n]}{k} \times \mathcal{SI}(n-k,2)$ then $S = T$ and so $\sigma = \tau$. □

This allows us, given N_n^2, to compute easily the numbers $SI(n,2)$, and $SR(n,2)$ for small values of n. Specifically one obtains that $SI(3,2) = 566$,

$SI(4,2) = 5769237$, and that $SR(3,2) = 10$, $SR(4,2) = 2283$.

11.2 Lower bound

In this subsection we obtain a lower bound for the number of stable permutations of rank 1 in $S([n]^2)$ and so, in particular, for the number of stable permutations.

Given $u \in S([n]^2)$ we find it convenient to let

$$F(u) := \{(a,b) \in [n]^2 : u(a,b) = (a,b)\}$$

so $F(u)$ is the set of fixed points of u.

For $n \in \mathbb{N}$ we let

$$D_n := \{u \in S([n]^2) : u_1(x,y) \neq u_2(a,b) \text{ for all } (x,y),(a,b) \notin F(u)\}.$$

Then, by Proposition 5.16, we have that the number of stable permutations of rank 1 is bounded from below by $|D_n|$. Thus, in particular, $N_n^2 \geq |D_n|$ for all $n \in \mathbb{N}$. This easily implies the following explicit lower bound.

Proposition 11.2. *Let $n \in \mathbb{N}$. Then $N_n^2 \geq 2(\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil)! - 1$.*

Proof. This follows from the observation that $S([\lfloor \frac{n}{2} \rfloor] \times [\lfloor \frac{n}{2} \rfloor + 1, n]) \subseteq D_n$, and similarly that $S([\lfloor \frac{n}{2} \rfloor + 1, n] \times [\lfloor \frac{n}{2} \rfloor]) \subseteq D_n$, and these two sets have only the identity permutation in common. $\square$

In particular, we obtain that $N_3^2 \geq 3$, $N_4^2 \geq 47$. These figures are lower than those obtained in the paragraph after Corollary 8.4, however they are greater for $n > 7$.

While the lower bound given by the previous proposition is very explicit, we now compute $|D_n|$ exactly.

We begin with the following observation, whose simple verification is omitted.

Lemma 11.3. *Let $n \in \mathbb{N}$. Then*

$$D_n = \{u \in S([n]^2) \mid a \neq y \textit{ for all } (x,y),(a,b) \notin F(u)\}.$$

Definition 11.4. For $u \in S([n^2])$ set

$$R(u) := \{i \in [n] \mid \exists j \in [n] : (i,j) \notin F(u)\}$$

$$C(u) := \{j \in [n] \mid \exists i \in [n] : (i,j) \notin F(u)\} \;,$$

so $R(u)$, resp. $C(u)$, is the set of rows, resp. columns, containing at least one element that is $\notin F(u)$. We observe that, for $u \in S([n]^2)$, $u \in D_n$ if and only if $R(u) \cap C(u) = \emptyset$. Furthermore, it is easy to see that $R({}^t u) = C(u)$ and $C({}^t u) = R(u)$.

Now,

$$D_n = \biguplus_{\{(I,J): I,J \subseteq [n], I \cap J = \emptyset\}} \{u \in S([n]^2) \ : \ R(u) = I, C(u) = J\}$$

(disjoint union). We are thus led to compute $|\mathcal{L}_{I,J}|$, where

$$\mathcal{L}_{I,J} := \{u \in S([n]^2) \ : \ R(u) = I, C(u) = J\}$$

(in order to simplify the notation we omit to write the index n). Observe that, for $I, J \neq \emptyset$, $\mathcal{L}_{\emptyset,J} = \emptyset = \mathcal{L}_{I,\emptyset}$, while $\mathcal{L}_{\emptyset,\emptyset} = \{e\}$. Moreover, $|\mathcal{L}_{I,J}| = |\mathcal{L}_{J,I}|$.

Lemma 11.5. *If $I, J, K, L \subseteq [n]$, $I \cap J = \emptyset$, $K \cap L = \emptyset$, $|I| = |K|$, $|J| = |L|$ then*

$$|\mathcal{L}_{I,J}| = |\mathcal{L}_{K,L}| \ .$$

Proof. Let $\{i_1, \dots, i_s\}_< := I$, $\{i_{s+1}, \dots, i_n\}_< := [n] \setminus I$. Let $\sigma_I \in S_n$ be defined by $\sigma_I(i_t) := t$ for all $t \in [n]$, and define σ_J similarly. Then the map $u \mapsto (\sigma_I \otimes \sigma_J) u (\sigma_I \otimes \sigma_J)^{-1}$ is a bijection between $\mathcal{L}_{I,J}$ and $\mathcal{L}_{[s],[r]}$, where $r := |J|$. □

Given $i, j \in [n]$ we therefore let

$$\mathcal{L}_{i,j} := |\mathcal{L}_{I,J}|$$

for any $I, J \subseteq [n]$, such that $I \cap J = \emptyset$, $|I| = i$, $|J| = j$.

Therefore,

$$\begin{aligned} |D_n| &= \sum_{\{(I,J):\, I,J \subset [n], I \cap J = \emptyset\}} |\mathcal{L}_{I,J}| \\ &= \sum_{i=0}^{n} \sum_{I \in \binom{[n]}{i}} \sum_{J \subseteq [n] \setminus I} |\mathcal{L}_{I,J}| \\ &= \sum_{i=0}^{n} \sum_{I \in \binom{[n]}{i}} \sum_{j=0}^{n-i} \sum_{J \in \binom{[n] \setminus I}{j}} |\mathcal{L}_{I,J}| \\ &= \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} \mathcal{L}_{i,j} \ . \end{aligned} \tag{19}$$

Given $a, b \in [n]$, we now wish to compute $\mathcal{L}_{a,b}$, which is also the cardinality of $\tilde{\mathcal{L}}_{a,b} = \{u \in S([a] \times [b]) \mid R(u) = [a], C(u) = [b]\}$.

Definition 11.6. For $u \in S([a] \times [b])$, let $M(u) \in \mathrm{Mat}_{a,b}(\{0,1\})$ be the $a \times b$-matrix defined by setting, for all $(i,j) \in [a] \times [b]$, $M(u)_{ij} = 1$ if $(i,j) \notin F(u)$ and $M(u)_{ij} = 0$ otherwise.

We observe that, for $u \in \tilde{\mathcal{L}}_{a,b}$, $\sum_{j=1}^{b} M(u)_{ij} > 0$ for all $i \in [a]$ and $\sum_{i=1}^{a} M(u)_{ij} > 0$ for all $j \in [b]$. Thus, letting

$$\mathrm{Mat}^*_{a,b} = \left\{ M \in \mathrm{Mat}_{a,b}(\{0,1\}) \ : \ \sum_{i=1}^{a} M_{ij} > 0 \ \forall j \in [b], \sum_{j=1}^{b} M_{ij} > 0 \ \forall i \in [a] \right\} ,$$

we have that $M(u) \in \mathrm{Mat}^*_{a,b}$ and hence

$$\mathcal{L}_{a,b} = \sum_{M \in \mathrm{Mat}^*_{a,b}} |\{u \in \tilde{\mathcal{L}}_{a,b} \mid M(u) = M\}| .$$

Although we will not use this, it is worth noting that $\mathrm{Mat}^*_{a,b}$ is the disjoint union of subsets determined by given row and column sum vectors, namely $\mathrm{Mat}_{a,b}(\lambda, \mu) := \left\{ M \in \mathrm{Mat}_{a,b}(\{0,1\}) \ : \ \sum_{i=1}^{a} M_{ij} = \lambda_j \ \forall j \in [b], \sum_{j=1}^{b} M_{ij} = \mu_i \ \forall i \in [a] \right\}$, where $\lambda = (\lambda_1, \ldots, \lambda_b) \in \mathbb{N}^b$, $\mu = (\mu_1, \ldots, \mu_a) \in \mathbb{N}^a$. The cardinalities of these subsets are invariant under permutation of the components of the row, resp. column vector and are given by the transition matrix between the elementary and monomial bases of the ring of symmetric functions, see for instance [24, Chapter I, section 6].

We remark that, for $M \in \mathrm{Mat}^*_{a,b}$,

$$\{u \in \tilde{\mathcal{L}}_{a,b} \ : \ M(u) = M\} = \{u \in S([a] \times [b]) \ : \ [n]^2 \setminus F(u) = S(M)\}$$

where

$$S(M) := \{(i,j) \in [a] \times [b] \ : \ M_{ij} = 1\} .$$

We now need to recall a well-known combinatorial notion. Given a set T, let $u \in S(T)$.

Definition 11.7. We say that u is a *derangement* if $u(t) \neq t$ for all $t \in T$.

Summing up, we have thus obtained the following result.

Proposition 11.8. *Let* $M \in \mathrm{Mat}^*_{a,b}$*. Then*

$$|\{u \in \tilde{\mathcal{L}}_{a,b} \ : \ M(u) = M\}| = d_{|S(M)|} ,$$

where $d_n := |\{\sigma \in S_n \mid \sigma \text{ is a derangement }\}|$.

It is perhaps worth recalling that $d_n \approx n!/e$, for $n \to \infty$ (see, e.g., [4, Theorem A, Section 4.2]).

All in all, we have thus arrived at the formula

$$\mathcal{L}_{a,b} = \sum_{M \in \mathrm{Mat}^*_{a,b}} d_{|S(M)|} \ . \tag{20}$$

Note that at this stage it is already possible to derive from formula (19) a lower bound for $|D_n|$ and thus for N_n^2 that is better than the one obtained in Proposition 11.2.

Proposition 11.9. *Let $n \in \mathbb{N}$, then*

$$N_n^2 \geq \sum_{i=0}^{n} \sum_{j=0}^{n-i} \binom{n}{i} \binom{n-i}{j} d_{ij} \ .$$

In particular, $N_n^2 \geq \binom{n}{\lfloor \frac{n}{2} \rfloor} d_{\lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil}$.

Proof. The first claim follows from (19) and (20). Taking the term in the sum with $i = \lfloor \frac{n}{2} \rfloor$ and $j = \lceil \frac{n}{2} \rceil$, the second bound follows. □

Using Stirling's formula, it is easy to see that $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is asymptotic to $\frac{2^n}{n^{1/2}} \sqrt{\frac{2}{\pi}}$ for $n \to \infty$.

Let $P_{a,b} \in \mathbb{Z}[q]$ be given by

$$P_{a,b}(q) := \sum_{M \in \mathrm{Mat}^*_{a,b}} q^{\|M\|} \ ,$$

where $\|M\| := \sum_{i=1}^{a} \sum_{j=1}^{b} M_{ij}$. Notice that for $M \in \mathrm{Mat}^*_{a,b}$ we have that $\|M\| = |S(M)|$. Moreover, one has that

$$P_{a,b}(q)|_{q^k \to d_k} = \mathcal{L}_{a,b} \ , \tag{21}$$

where $P_{a,b}(q)|_{q^k \to d_k}$ means that, in the expression of $P_{a,b}$ as a sum of monomials, we replace each q^k with d_k for all k. Hence, it suffices to compute $P_{a,b}(q)$.

Let, for brevity

$$\overline{\mathrm{Mat}}_{a,b}(\{0,1\}) := \{M \in \mathrm{Mat}_{a,b}(0,1) \ : \ \sum_{i=1}^{a} M_{ij} > 0 \ \forall j \in [b]\}.$$

For $T \subseteq [a]$ we let

$$f(T) := \sum_{\{M \in \overline{\mathrm{Mat}}_{a,b}(\{0,1\}) \,:\, \sum_j M_{lj}=0 \text{ if } l \in T\}} q^{\|M\|} ,$$

and

$$g(T) := \sum_{\{M \in \overline{\mathrm{Mat}}_{a,b}(\{0,1\}) \,:\, \sum_j M_{lj}=0 \text{ if and only if } l \in T\}} q^{\|M\|} .$$

Note that $f(T) = \sum_{[a]\supseteq S \supseteq T} g(S)$ for all $T \subseteq [a]$, and that $g(\emptyset) = P_{a,b}(q)$. Hence, by the Principle of Inclusion-Exclusion, we have that

$$P_{a,b}(q) = \sum_{S \subseteq [a]} (-1)^{|S|} f(S).$$

On the other hand,

$$f(S) = \left((1+q)^{a-|S|} - 1\right)^b .$$

Therefore,

$$\begin{aligned} P_{a,b}(q) &= \sum_{S \subseteq [a]} (-1)^{|S|} \left((1+q)^{a-|S|} - 1\right)^b \\ &= \sum_{j=0}^{a} (-1)^j \left((1+q)^{a-j} - 1\right)^b \binom{a}{j} \\ &= \sum_{j=0}^{a} \sum_{k=0}^{b} (-1)^{b+k+j} \binom{a}{j} \binom{b}{k} (1+q)^{k(a-j)}. \end{aligned}$$

We now note the following useful fact.

Lemma 11.10. *Let $t \in \mathbb{N}$. Then*

$$(1+q)^t|_{q^k \to d_k} = t! .$$

Proof. It is easy to see that

$$(1+q)^t|_{q^k \to d_k} = \sum_{k=0}^{t} \binom{t}{k} d_k = t! .$$

□

By the last lemma we therefore conclude that

$$\mathcal{L}_{a,b} = P_{a,b}(q)|_{q^k \to d_k} = \sum_{j=0}^{a} \sum_{k=0}^{b} (-1)^{b+k+j} \binom{a}{j} \binom{b}{k} \big(k(a-j)\big)! \tag{22}$$

Concluding, we have the following formula for D_n.

Theorem 11.11. *Let $n \in \mathbb{N}$, then*

$$|D_n| = \sum_{a=0}^{n} \sum_{b=0}^{n-a} \sum_{j=0}^{a} \sum_{k=0}^{b} (-1)^{b+k+a+j} \binom{n}{j, a-j, k, b-k, n-a-b} (kj)! \ .$$

Proof. By (19), and (22) we have that

$$\begin{aligned} |D_n| &= \sum_{a=0}^{n} \sum_{b=0}^{n-a} \binom{n}{a} \binom{n-a}{b} \sum_{j=0}^{a} \sum_{k=0}^{b} (-1)^{b+k+j} \binom{a}{j} \binom{b}{k} \big(k(a-j)\big)! \\ &= \sum_{a=0}^{n} \sum_{b=0}^{n-a} \sum_{j=0}^{a} \sum_{k=0}^{b} (-1)^{b+k+a+j} \binom{a}{j} \binom{b}{k} \binom{n}{a} \binom{n-a}{b} (kj)! \end{aligned}$$

and the result follows. □

One can check that $D_1 = D_2 = 1$, $D_3 = 7$, $D_4 = 155$, $D_5 = 13781$, $D_6 = 8383469$, $D_7 = 33623552299$. The seven permutations counted by D_3 are the identity and the 6 transpositions that satisfy the equivalent conditions of Theorem 8.1.

As stated at the beginning of this subsection, $|D_n|$ provides a lower bound for N_n^2. Therefore, we get the following bound that improves the one obtained in Proposition 11.9.

Corollary 11.12. *Let $n \in \mathbb{N}$. Then*

$$N_n^2 \geq \sum_{a=0}^{n} \sum_{b=0}^{n-a} \sum_{j=0}^{a} \sum_{k=0}^{b} (-1)^{b+k+a+j} \binom{n}{j, a-j, k, b-k, n-a-b} (kj)! \ .$$

Using the Bonferroni inequalities (see, e.g., [4, Section 4.7]), one can show that

$$\mathcal{L}_{a,b} \geq \sum_{i=0}^{b} \binom{b}{i} (-1)^{b-i} \left[(ai)! - a[(ai-i)!]\right]$$

for all $a, b \in \mathbb{N}$ and from this deduce, after some work, that

$$\mathcal{L}_{a,b} \geq (ab)! - a[(b(a-1))!] - b[(a(b-1))!] + ab[((a-1)(b-1))!] \ .$$

However, we feel that this does not lead to an immediate significant improvement over Proposition 11.9.

11.3 Upper bound

Given that we obtained a lower bound for the number of stable permutations, it is natural to look for some upper bound. This of course entails finding conditions for a permutation not to be stable. In this subsection we give one such condition and compute the corresponding upper bound.

Proposition 11.13. *Let $a, b, c, d, i \in [n]$, $a \neq b$, and $u \in S([n]^2)$ be such that $u(i, a) = (a, b)$, $u^{-1}(i, a) = (c, b)$ and $u^{-1}(b, b) = (d, b)$. Then u is not stable.*

Proof. Let $k \in \mathbb{N}$. Then one can compute that

$$\psi_k(u)(\underbrace{i, i, \ldots, i}_{k+1}, a) = (c, \underbrace{d, d, \ldots, d}_{k}, b),$$

so u is not stable. □

From the previous proposition we deduce the following (certainly improvable) lower bound on the number of permutations that are not stable.

Corollary 11.14. *In $S([n]^2)$ there are at least $4((n^2-3)!)$ permutations that are not stable.*

Proof. This follows by taking $d = b = 1$, $c = i = a = 2$, $d = a = 1$, $c = i = b = 2$, $b = d = i = 1$, $c = a = 2$ and $d = a = i = 1$, $c = b = 2$ in Proposition 11.13. □

12 Directions for further work

Without any attempt at being exhaustive, in this final section we collect a few problems and conjectures arising from this work.

From the point of view of the explicit characterization of various classes of stable permutations we feel that the following is natural.

Conjecture 12.1. *Let $(a_1, b_1), \ldots, (a_k, b_k) \in [n]^2$ be distinct. Then*

$$\big((a_1, b_1), (a_2, b_2), \ldots, (a_k, b_k)\big)$$

is stable of rank 1 if and only if $\{a_1, \ldots, a_k\} \cap \{b_1, \ldots, b_k\} = \emptyset$.

The above conjecture holds if $k \leq 3$ by Theorems 8.1 and 8.7, and it holds if $a_1, \ldots, a_k$ (or $b_1, \ldots, b_k$) are distinct by Propositions 8.5 and 8.6. Note that the enumeration of stable k-cycles of rank 1 in $S([n]^2)$ for all $n \in \mathbb{N}$ and $k \geq 4$ would follow from Conjecture 12.1.

Another natural family of permutations to analyze is involutions. This seems to be a much harder problem. Perhaps the most natural class of permutations to consider next would be the product of two commuting transpositions. We leave an exhaustive analysis of this case to future work. Here we limit ourselves to observing that we know from Proposition 5.6 and Theorem 5.2 that the product of a horizontal and a vertical stable transpositions (in this order) is stable. For the case of two horizontal stable transpositions we have the following conjecture.

Conjecture 12.2. *Let $(a_1, b_1), (a_1, b_2), (a_2, b_3), (a_2, b_4) \in [n]^2$ be distinct, $a_1 \neq a_2$. Then*

$$u := \big((a_1, b_1), (a_1, b_2)\big)\, \big((a_2, b_3), (a_2, b_4)\big)$$

is stable of rank 1 if and only if one of the following conditions is satisfied:

(i) $\{a_1, a_2\} \cap \{b_1, b_2, b_3, b_4\} = \emptyset$*;*

(ii) $\{a_1, a_2\} = \{b_1, b_2, b_3, b_4\}$*.*

We have verified that Conjecture 12.2 holds if $n \leq 8$. Also, note that if condition (i) holds then, by Corollary 5.16, u is stable of rank 1. Further, one can check that if condition (ii) holds then without loss of generality we may assume that $a_1 = b_1 = b_3$ and $a_2 = b_2 = b_4$, and in this case it is not hard to show that u is again stable of rank 1. Therefore, Conjecture 12.2 holds if one can show that either (i) or (ii) is a necessary condition for u to be stable of rank 1.

From the enumerative point of view the most fundamental problem is definitely the following.

Problem 12.3. *Determine the numbers N_n^r of stable permutations in $S([n]^r)$ for all values of n and r.*

Note that it is known that N_n^r is always divisible by $n^{r-1}!$, see [9, Section 6].

A natural and possibly easier problem is that of enumerating the stable permutations u in $S([n]^2)$ (or in $S([n]^r)$) of rank 1 or those such that λ_u is an involution. For $r = 2$, by Propositions 4.5 and 4.11, this is equivalent to the following.

Problem 12.4. *Compute*

$$|\{u \in S([n]^2) : \ (1 \otimes u)(u \otimes 1) = (u \otimes 1)(1 \otimes u)\}|$$

and

$$|\{u \in S([n]^2) \ : \ (1 \otimes u^{-1})(u^{-1} \otimes 1)(1 \otimes u) = (u \otimes 1)\}|$$

for all $n \in \mathbb{N}$.

In the case of permutations in $S_n \otimes S_n$ these problems can be solved. Indeed, by Proposition 4.10 we have that

$$|\{u \in S_n \otimes S_n : \ (1 \otimes u)(u \otimes 1) = (u \otimes 1)(1 \otimes u)\}| = |\{(x, y) \in S_n \times S_n \ : \ xy = yx\}|$$

and this number is known to be (see, e.g., [29, Ex. 5.12]) $p(n)\, n!$ where $p(n)$ is the number of partitions of n. Similarly, by Proposition 4.12 we have that

$$\begin{aligned} &|\{u \in S_n \otimes S_n : \ (1 \otimes u^{-1})(u^{-1} \otimes 1)(1 \otimes u) = (u \otimes 1)\}| \\ &\qquad = |\{(x, y) \in S_n \times S_n \ : \ x^2 = (yx)^2 = 1\}| \end{aligned}$$

so this number is t_n^2 where t_n is the number of involutions in S_n, or equivalently, the number of standard Young tableaux of size n (we refer the reader to, e.g., [29], for further information about these numbers).

The enumeration of stable permutations by rank is also a natural problem. This distribution is only known for $n \leq 3$ and $r = 2$, in which case it is 4 and $54 + 186x + 240x^2 + 96x^3$ (where the coefficient of x^k is the number of stable permutations in $S([n]^2)$ of rank $k+1$). A related problem (see Proposition 4.4) is that of computing

$$E_k(n) := |\{u \in S([n]^2) : \psi_k(u) \in S([n]^{k+1}) \otimes 1\}|$$

for all $n \geq 2$ and $k \geq 1$. In the case of $n = 3$ one can check that $E_1(3) = 144$, $E_2(3) - E_1(3) = 288$, and $E_3(3) - E_2(3) = 144$, while $E_k(3) = 576$ for all $k \geq 4$.

We feel, and the empirical evidence suggests, that stable permutations in $S([n]^r)$ are extremely rare. More precisely, we believe that the following holds.

Conjecture 12.5.

$$\lim_{r \to \infty} \frac{N_n^r}{n^r!} = 0 \ , \textit{for all } n \geq 2 \ ,$$

$$\lim_{n \to \infty} \frac{N_n^r}{n^r!} = 0 \ , \textit{for all } r \geq 2 \ .$$

Let T_n^2 denote the number of permutations in $S([n]^2)$ obtained as compatible products of stable transpositions.

Problem 12.6. *Does the limit* $\lim_{n\to\infty} \frac{T_n^2(n!)^2}{N_n^2}$ *exist?*

From a group theoretical point of view, we feel that the finding a generating set for the reduced Weyl group of $\mathcal{O}_n$ is a fundamental problem. It is of course well known that finite symmetric groups are generated by transpositions. We have characterized stable transpositions in $S([n]^2)$ and we expect that it should be possible to extend this to $S([n]^r)$, $r \in \mathbb{N}$. In this respect, the following seem like natural questions to investigate.

Problem 12.7. *Let* $n \in \mathbb{N}$. *Is the reduced Weyl group of* $\mathcal{O}_n$ *generated by*

$$\Big\{\lambda_t : t \in \bigcup_{r\in\mathbb{N}} S([n]^r),\ t \text{ stable transposition}\Big\}\ ?$$

Problem 12.8. *Let* $n \in \mathbb{N}$. *Is the reduced Weyl group of* $\mathcal{O}_n$ *generated by involutions?*

Of course, the same questions would also make sense for the image of the reduced Weyl group in $\mathrm{Out}(\mathcal{O}_n)$ under the quotient map.

In light of Theorem 2.1 it is interesting to investigate, more generally, which unitary matrices $U \in U_{n^r}(\mathbb{C})$ yield automorphisms of the Cuntz algebra $\mathcal{O}_n$. In particular, we feel that examining which elements of the complex reflection group $G(k,n)$ (also known as the group of colored permutations) give rise to automorphisms of $\mathcal{O}_n$ is a natural and interesting avenue of further research, which could give rise to arithmetic identities. In this direction the case $k=2$ (i.e., the Weyl group of type B_n) would be a natural starting point.

It is known (see [11, Proposition 1.5]) that every stable permutation also gives rise to an homeomorphism of the Cantor set. However, this condition is strictly weaker than that of being stable, and has been characterized in [9, Theorem 3.4]. It would be interesting to study also these permutations from a combinatorial point of view.

We plan to investigate these questions in a future work.

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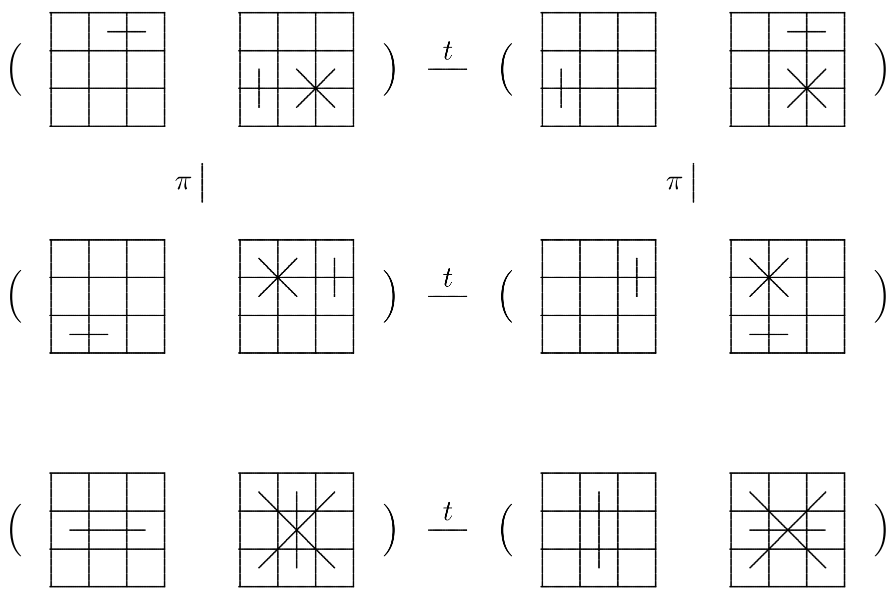
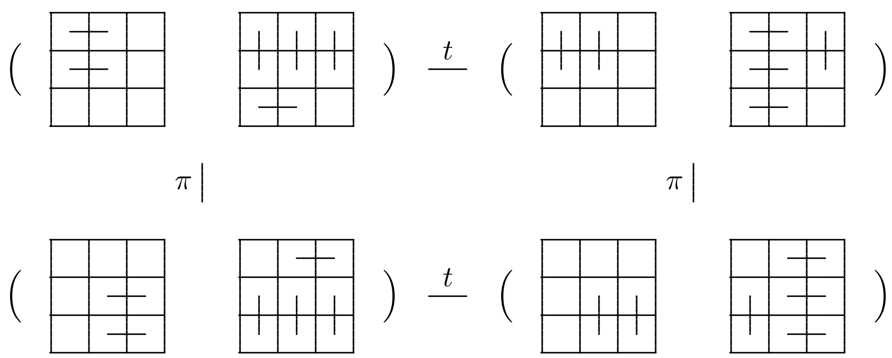
13 Appendix: some figures

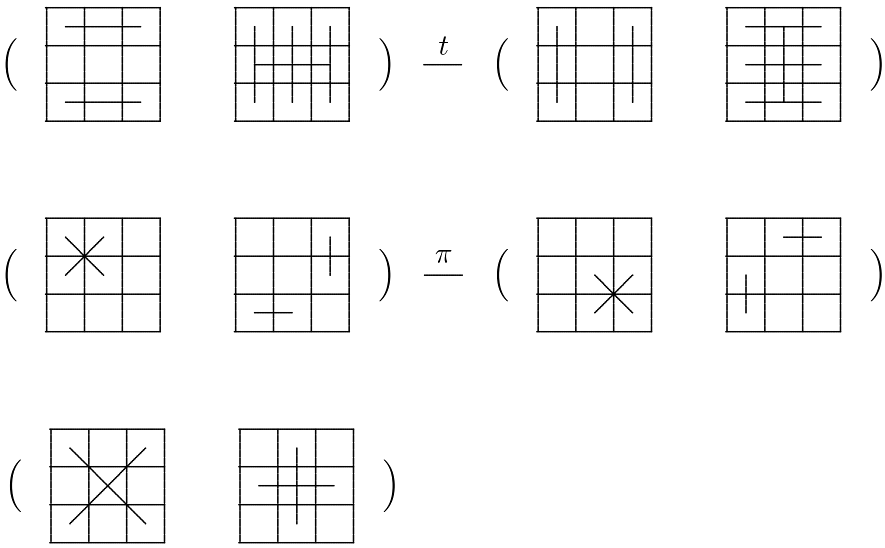
Figure 1: the 36 permutations $u \in S([3]^2) \setminus S([3]) \otimes S([3])$ such that $(\lambda_u)^{-1} = \lambda_u$, and their symmetries.

Simple transpositions:



Immersions:





Triple transpositions:

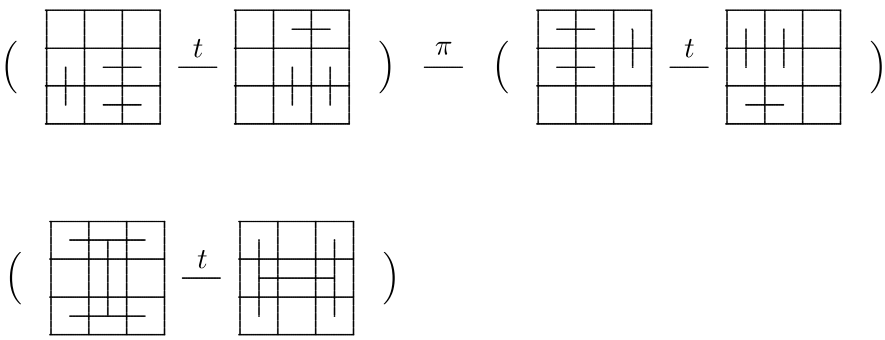


Figure 2: the 15 non-trivial representatives of the 16 double orbits of the 576 stable permutations of $S([3]^2)$, with their symmetries.