Convergence of fractional diffusion processes in extension domains

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Abstract

We study the asymptotic behavior of anomalous fractional diffusion processes in bad domains via the convergence of the associated energy forms. We introduce the associated Robin-Venttsel' problems for the regional fractional Laplacian. We provide a suitable notion of fractional normal derivative on irregular sets via a fractional Green formula as well as existence and uniqueness results for the solution of the Robin-Venttsel' problem by a semigroup approach. Submarkovianity and ultracontractivity properties of the associated semigroup are proved.

Keywords: Fractional Laplacian, fractal domains, fractional Green formula, M-convergence, semigroups, dynamical boundary conditions.

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Introduction

In this paper we study a heat equation for the regional fractional Laplacian with Robin (Venttsel') boundary conditions in irregular domains (e.g. Jones domains [20]). This type of problems belongs to the large class of anomalous diffusion processes. In the recent years there has been an increasing interest in their study due to the different application fields.

The anomalous diffusion is an important topic in physics, finance and probability ([1, 19, 36, 38]; for a tutorial see [40]). Mathematically it is described by a nonlocal operator. Several models appear in the literature to describe such diffusion, e.g. the fractional Brownian motion, the continuous time random walk, the Lévy flight as well as random walk models based on evolution equations of single and distributed fractional order in time and/or space [8, 15, 35, 38, 39].
In these different frameworks, if the regional fractional Laplacian is considered (see (2.1)), the corresponding diffusion processes take place across irregular interfaces or boundaries, possibly of fractal type. A key point is to give a rigorous mathematical formulation for these processes as well as to study their "smoother approximations" in view of concrete numerical simulations.
In the literature, results for boundary value problems for the regional fractional Laplacian with Dirichlet, Neumann and more general non-standard boundary conditions, such as dynamical boundary conditions of Venttsel' or Robin type for piecewise smooth (Lipschitz) domains, can be found in [12], [13] and [14] along with the physical motivations.
To our knowledge, the case of Robin-Venttsel' problems for the regional fractional Laplacian in irregular domains (studied e.g. for second order elliptic operators in divergence form in [33]) is here investigated for the first time.
Our aim, in this paper, is twofold; the former is to give a rigorous formulation of a parabolic problem for the regional fractional Laplacian with dynamical boundary conditions in irregular domains and in suitable smoother approximating domains. The latter is to prove that the approximating processes converge in a suitable sense to the diffusion process of the irregular case.
More precisely, in this paper we consider the following evolution problems for the regional fractional Laplacian with dynamical Robin-Venttsel' boundary conditions in an irregular domain Q as well as in the corresponding approximating domains Q_n.
The problems can be formally stated as:

$$(\tilde{P}) \begin{cases} \frac{\partial u}{\partial t}(t,x) + (-\Delta)^s_Q u(t,x) = f(t,x) & \text{in } (0,T]\times Q, \\ \frac{\partial (u|_{\partial Q})}{\partial t} + \mathcal{N}_{2-2s} u + b u|_{\partial Q} = f & \text{on } (0,T]\times\partial Q, \\ u(0,x) = u_0(x) & \text{in } \overline{Q}, \end{cases}$$

and, for every $n \in \mathbb{N}$,

$$
(\tilde{P}_n) \begin{cases} \frac{\partial u_n}{\partial t}(t,x) + (-\Delta)^s_{Q_n} u_n(t,x) = f_n(t,x) & \text{in } (0,T] \times Q_n, \\ \delta_n \frac{\partial (u_n|_{\partial Q_n})}{\partial t} + \mathcal{N}_{2-2s} u_n + \delta_n b u_n|_{\partial Q_n} = \delta_n f_n & \text{on } (0,T] \times \partial Q_n, \\ u_n(0,x) = u_0^{(n)}(x) & \text{in } \overline{Q}_n, \end{cases}
$$

where Q is a bounded (ε,δ) domain having as boundary a d-set (see Definitions 1.1 and 1.2) possibly of fractal type and $\{Q_n\}$ is a sequence of suitable smooth domains approximating Q. Here $(-\Delta)^s_Q$ and $(-\Delta)^s_{Q_n}$ denote the regional fractional Laplacians (see (2.1)), $s \in (0,1)$, $\mathcal{N}_{2-2s}u$ is the fractional normal derivative to be suitably defined, f, f_n, b, u_0 and $u_0^{(n)}$ are given functions, while T and δ_n are positive numbers.
We introduce a suitable notion of fractional normal derivative on irregular sets, via a generalized fractional Green formula, and we prove that it is an element of the dual of a suitable Besov space defined on ∂Q (see Theorem 2.2).
We consider the fractional energy form E_s defined in (3.2), which turns out to be a closed nonlocal Dirichlet form in $L^2(Q,m)$ (see (3.1)), and the corresponding associated generator A_s. In Theorem 4.1 we prove, via a semigroup approach, existence and uniqueness of a classical solution for a suitable abstract Cauchy problem (P) for the operator A_s. We prove regularity properties of the semigroup, i.e. markovianity, order preserving and ultracontractivity in Theorems 3.5 and 3.6. In Theorem 4.2 we prove that problem $(\tilde{P})$ is the strong formulation of the abstract problem (P). Similar results for the approximating problems $(\tilde{P}_n)$ hold.
In order to study the asymptotic behavior of the approximating solutions, we consider the case of a three-dimensional Koch-type cylinder Q and its corresponding polyhedral approximating domains Q_n. We consider the fractional energy forms E_s and $E_s^{(n)}$ on $L^2(Q,m)$ and $L^2(Q,m_n)$ respectively (see (5.4) and (5.7)). In the pre-fractal case, existence and uniqueness of a classical solution of the associated abstract Cauchy problem (P_n) as well as a strong interpretation are given respectively in Theorems 5.2 and 5.3. In the fractal case, these results follow from Theorems 4.1 and 4.2 specialized to this case. In Theorem 7.3 we study the asymptotic behavior of the solutions of problems (P_n); the functional setting is that of varying Hilbert spaces (see Section 1.2). The Mosco-Kuwae-Shioya convergence of the fractional energy forms, proved in Theorem 6.5, and that of semigroups, given in Theorem 6.6, yield the convergence of the solutions in a suitable sense. The choice of the factor δ_n, which accounts for the jump of dimension between ∂Q and ∂Q_n, is crucial in the proof of the energy convergence.
Finally, in Theorems 7.6 and 7.9, we prove the convergence of the time derivatives and the convergence of the fractional normal derivatives in a suitable weak sense.
The plan of the paper is the following.

In Section 1, we recall some preliminary results on traces and varying Hilbert spaces.
In Section 2, we recall the definition of fractional regional Laplacian and we introduce the notion of weak fractional normal derivative by proving a generalized fractional Green formula for irregular domains.
In Section 3 we introduce the nonlocal fractional energy form on Q. We prove that it is closed, symmetric and Markovian and that the associated semigroup is positive preserving, L^∞-contractive and ultracontractive.
In Section 4 we prove existence and uniqueness of a classical solution for the corresponding abstract Cauchy problem and we give a strong interpretation.
In Section 5 we consider as Q a Koch-type cylinder, i.e. $Q = \Omega \times [0,1]$, where Ω is the snowflake domain. We introduce the corresponding approximating domains Q_n, the corresponding fractional energy forms $E_s^{(n)}$ defined in (5.6), the associated semigroups and Cauchy problems.
In Section 6 we recall the notion of Mosco-Kuwae-Shioya convergence of the forms and we prove the M-convergence of the energy forms $E_s^{(n)}$ to E_s.
In Section 7 we prove the convergence of the solutions in the framework of varying Hilbert spaces as well as the convergence of the time derivatives. The convergence of the fractional normal derivative is then achieved in a suitable weak sense.

1 Preliminaries

1.1 Functional spaces and trace theorems

Let $\mathcal{G}$ (resp. $\mathcal{S}$) be an open (resp. closed) set of $\mathbb{R}^N$. By $L^2(\mathcal{G})$ we denote the Lebesgue space with respect to the Lebesgue measure $\mathrm{d}\mathcal{L}_N$, which will be left to the context whenever that does not create ambiguity. By $L^p(\partial\mathcal{G}, \mu)$ we denote the Lebesgue space on $\partial\mathcal{G}$ with respect to a Hausdorff measure μ supported on $\partial\mathcal{G}$. By $D(\mathcal{G})$ we denote the space of infinitely differentiable functions with compact support on $\mathcal{G}$. By $C(\mathcal{S})$ we denote the space of continuous functions on $\mathcal{S}$.
By $H^s(\mathcal{G})$, where $0 < s < 1$, we denote the fractional Sobolev space of exponent s. Endowed with the following norm

$$\|u\|^2_{H^s(\mathcal{G})} = \|u\|^2_{L^2(\mathcal{G})} + \iint_{\mathcal{G}\times\mathcal{G}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y),$$

it becomes a Banach space. We denote by $|u|_{H^s(\mathcal{G})}$ the seminorm associated to $\|u\|_{H^s(\mathcal{G})}$ and by $(u,v)_{H^s(\mathcal{G})}$ the scalar product induced by the H^s-norm. Moreover, we set

$$(u,v)_s := \iint_{\mathcal{G}\times\mathcal{G}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

In the following we will denote by $|A|$ the Lebesgue measure of a measurable subset $A \subset \mathbb{R}^N$. For f in $H^s(\mathcal{G})$, we define the trace operator γ_0 as

$$\gamma_0 f(x) := \lim_{r \to 0} \frac{1}{|B(x,r) \cap \mathcal{G}|} \int\limits_{B(x,r)\cap\mathcal{G}} f(y)\, \mathrm{d}\mathcal{L}_N(y) \tag{1.1}$$

at every point $x \in \overline{\mathcal{G}}$ where the limit exists. The limit (1.1) exists at quasi every $x \in \overline{\mathcal{G}}$ with respect to the $(s,2)$-capacity (see [2], Defnition 2.2.4 and Theorem 6.2.1 page 159). In the sequel we will omit the trace symbol and the interpretation will be left to the context.

We now recall the definition of (ε, δ) domain. For details see [20].

Definition 1.1. Let $\mathcal{F} \subset \mathbb{R}^N$ be open and connected. For $x \in \mathcal{F}$, let $d(x) := \inf\limits_{y \in \mathcal{F}^c} |x - y|$. We say that $\mathcal{F}$ is an (ε, δ) domain if, whenever $x, y \in \mathcal{F}$ with $|x-y| < \delta$, there exists a rectifiable arc $\gamma \in \mathcal{F}$ joining x to y such that

$$\ell(\gamma) \leq \frac{1}{\varepsilon}|x-y| \quad \text{ and } \quad d(z) \geq \frac{\varepsilon|x-z||y-z|}{|x-y|} \text{ for every } z \in \gamma.$$

In this paper, we consider two particular classes of (ε, δ) domains $Q \subset \mathbb{R}^N$. More precisely, Q can be a (ε, δ) domain having as boundary either a d-set or an arbitrary closed set in the sense of [21]. For the sake of completeness, we recall the definition of d-set given in [22].

Definition 1.2. A closed nonempty set $\mathcal{M} \subset \mathbb{R}^N$ is a d-set (for $0 < d \leq N$) if there exist a Borel measure μ with $\operatorname{supp} \mu = \mathcal{M}$ and two positive constants c_1 and c_2 such that

$$c_1 r^d \leq \mu(B(x,r) \cap \mathcal{M}) \leq c_2 r^d \quad \forall\, x \in \mathcal{M}. \tag{1.2}$$

The measure μ is called d-measure.

We recall the definition of Besov spaces on an arbitrary closed set $\tilde{\mathcal{F}}$ specialized to our case. For generalities on these Besov spaces, we refer to [21]. Let us suppose that there is a measure $\mu_{\tilde{\mathcal{F}}}$ on $\tilde{\mathcal{F}}$ satisfying the following condition: for $0 < d_1 \leq d_2 \leq N$, there exist two positive constants $\tilde{c}_1$ and $\tilde{c}_2$ such that

$$\tilde{c}_1\, k^{d_1} \mu_{\tilde{\mathcal{F}}}(B(x,r)) \leq \mu_{\tilde{\mathcal{F}}}(B(x,kr)) \leq \tilde{c}_2\, k^{d_2} \mu_{\tilde{\mathcal{F}}}(B(x,r)) \tag{1.3}$$

for all $x \in \tilde{\mathcal{F}}$, $r > 0$, $k \geq 1$ such that $kr \leq 1$.

Definition 1.3. Let $\tilde{\mathcal{F}} \subset \mathbb{R}^N$ be an arbitrary closed set and $\mu_{\tilde{\mathcal{F}}}$ be a measure defined on $\tilde{\mathcal{F}}$ satisfying (1.3). The Besov space $\tilde{B}^{2,2}_\gamma(\tilde{\mathcal{F}})$ with respect to $\mu_{\tilde{\mathcal{F}}}$ is the space of functions such that the following norm is finite:

$$|||u|||^2_{\tilde{B}^{2,2}_\gamma(\tilde{\mathcal{F}})} = \|u\|^2_{L^2(\tilde{\mathcal{F}})} + \sum_{j=0}^{+\infty} 3^{j(2\gamma-N)} \iint\limits_{|x-y|<3^{-j}} \frac{|u(x)-u(y)|^2}{m_j(x) m_j(y)}\, \mathrm{d}\mu_{\tilde{\mathcal{F}}}(x)\, \mathrm{d}\mu_{\tilde{\mathcal{F}}}(y), \tag{1.4}$$

where $m_j(x) := \mu_{\tilde{\mathcal{F}}}(B(x, 3^{-j}))$.

From Proposition 2 in [21], it follows that this norm is equivalent to the following norm:

$$\|u\|^2_{\tilde{B}^{2,2}_\gamma(\tilde{\mathcal{F}})} = \|u\|^2_{L^2(\tilde{\mathcal{F}})} + \iint\limits_{|x-y|<1} \frac{|u(x)-u(y)|^2}{|x-y|^{2\gamma-N}(\mu_{\tilde{\mathcal{F}}}(B(x,|x-y|)))^2} \,\mathrm{d}\mu_{\tilde{\mathcal{F}}}(x)\,\mathrm{d}\mu_{\tilde{\mathcal{F}}}(y). \quad (1.5)$$

For further purposes we state a trace theorem for functions in $H^s(Q)$ where Q is a bounded (ε, δ) domain with boundary ∂Q an arbitrary closed set satisfying (1.3) (see Theorem 1 in [21]).

Proposition 1.4. *Let $\frac{1}{2} < s < 1$. $\tilde{B}^{2,2}_s(\partial Q)$ is the trace space of $H^s(Q)$ in the following sense:*

(i) γ_0 is a continuous linear operator from $H^s(Q)$ to $\tilde{B}^{2,2}_s(\partial Q)$;

(ii) there exists a continuous linear operator Ext *from $\tilde{B}^{2,2}_s(\partial Q)$ to $H^s(Q)$ such that $\gamma_0 \circ \mathrm{Ext}$ is the identity operator in $\tilde{B}^{2,2}_s(\partial Q)$.*

By $(\tilde{B}^{2,2}_s(\partial Q))'$ we denote the dual space of $\tilde{B}^{2,2}_s(\partial Q)$, see [23].
In order for the trace to be well defined, from now on we suppose that

$$\frac{1}{2} < s < 1.$$

1.2 Varying Hilbert spaces

In this subsection, we introduce the notion of convergence of varying Hilbert spaces. We refer to [26] and [25] for definitions and proofs. The Hilbert spaces we consider are real and separable.

Definition 1.5. A sequence of Hilbert spaces $\{H_n\}_{n\in\mathbb{N}}$ converges to a Hilbert space H if there exists a dense subspace $C \subset H$ and a sequence $\{Z_n\}_{n\in\mathbb{N}}$ of linear operators $Z_n \colon C \subset H \to H_n$ such that

$$\lim_{n\to\infty} \|Z_n u\|_{H_n} = \|u\|_H \text{ for any } u \in C.$$

We set $\mathcal{H} = \{\cup_n H_n\} \cup H$ and define strong and weak convergence in $\mathcal{H}$. From now on we assume that $\{H_n\}_{n\in\mathbb{N}}$, H and $\{Z_n\}_{n\in\mathbb{N}}$ are as in Definition 1.5.

Definition 1.6 (Strong convergence in $\mathcal{H}$)**.** A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if $u_n \in H_n$, $u \in H$ and there exists a sequence $\{\widetilde{u}_m\}_{m\in\mathbb{N}} \in C$ tending to u in H such that

$$\lim_{m\to\infty} \overline{\lim_{n\to\infty}} \|Z_n \widetilde{u}_m - u_n\|_{H_n} = 0.$$

Definition 1.7 (Weak convergence in $\mathcal{H}$)**.** A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ weakly converges to u in $\mathcal{H}$ if $u_n \in H_n$, $u \in H$ and

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ strongly tending to v in $\mathcal{H}$.

We remark that the strong convergence implies the weak convergence (see [26]).

Lemma 1.8. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence weakly converging to u in $\mathcal{H}$. Then*

$$\sup_n \|u_n\|_{H_n} < \infty, \quad \|u\|_H \le \varliminf_{n\to\infty} \|u_n\|_{H_n}.$$

Moreover, $u_n \to u$ strongly if and only if $\|u\|_H = \lim_{n\to\infty} \|u_n\|_{H_n}$.

Let us recall some characterizations of the strong convergence of a sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$.

Lemma 1.9. *Let $u \in H$ and let $\{u_n\}_n \in \mathbb{N}$ be a sequence of vectors $u_n \in H_n$. Then $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in $\mathcal{H}$ if and only if*

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ with $v_n \in H_n$ weakly converging to a vector v in $\mathcal{H}$.

Lemma 1.10. *A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in H_n$ strongly converges to a vector u in $\mathcal{H}$ if and only if*

$$\begin{array}{rclll} \|u_n\|_{H_n} & \to & \|u\|_H & & \textit{and} \\ (u_n, Z_n(\varphi))_{H_n} & \to & (u, \varphi)_H & & \textit{for every } \varphi \in C. \end{array}$$

Lemma 1.11. *Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence with $u_n \in H_n$. If $\|u_n\|_{H_n}$ is uniformly bounded, then there exists a subsequence of $\{u_n\}_{n\in\mathbb{N}}$ which weakly converges in $\mathcal{H}$.*

Lemma 1.12. *For every $u \in H$ there exists a sequence $\{u_n\}_{n\in\mathbb{N}}$, with $u_n \in H_n$, strongly converging to u in $\mathcal{H}$.*

We now introduce the notion of strong convergence of operators. We denote by $\mathfrak{L}(X)$ the space of linear and continuous operators on a Hilbert space X.

Definition 1.13. A sequence of bounded operators $\{B_n\}_{n\in\mathbb{N}}$, with $B_n \in \mathfrak{L}(H_n)$, strongly converges to an operator $B \in \mathfrak{L}(H)$ if for every sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ with $u_n \in H_n$ strongly converging to a vector u in $\mathcal{H}$, the sequence $\{B_h u_h\}$ strongly converges to Bu in $\mathcal{H}$.

2 The regional fractional Laplacian and the Green formula

We introduce the so-called " regional" fractional Laplacian, for the defnition we refer to [3, 4, 16, 17, 18].
Let $s \in (0,1)$. We define the space

$$\mathcal{L}^1_s(\mathcal{G}) := \left\{ u\colon \mathcal{G} \to \mathbb{R} \text{ measurable } : \int\limits_{\mathcal{G}} \frac{|u(x)|}{(1+|x|)^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x) < \infty \right\}.$$

The regional fractional Laplacian $(-\Delta)^s_{\mathcal{G}}$ is defined as follows, for $x \in \mathcal{G}$:

$$(-\Delta)^s_{\mathcal{G}} u(x) = C_{N,s} \mathrm{P.V.} \int\limits_{\mathcal{G}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(y) = C_{N,s} \lim_{\varepsilon \to 0^+} \int\limits_{\{y \in \mathcal{G}\,:\,|x-y|>\varepsilon\}} \frac{u(x)-u(y)}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(y), \tag{2.1}$$

provided that the limit exists for every function $u \in \mathcal{L}^1_s(\mathcal{G})$. The positive constant $C_{N,s}$ is defined as follows:

$$C_{N,s} = \frac{s 2^{2s} \Gamma(\frac{N+2s}{2})}{\pi^{\frac{N}{2}} \Gamma(1-s)},$$

where Γ is the Euler function.
We refer to the Introductions of [12] and [14] for detailed discussions on the relation between the fractional Laplacian and the regional fractional Laplacian (see e.g. Section 2 in [12]).
We now give a suitable definition of fractional normal derivative on non-smooth domains. In the literature (see [14]) there are different definitions of fractional normal derivatives on Lipschitz domains. Our aim is to prove a fractional Green formula, inspired by Definition 2.9 (b) of [14], for (ε,δ) domains Q with fractal boundary. A key tool is the use of a limit argument. More precisely, Q is approximated by an increasing sequence of non-convex domains Q_n with Lipschitz boundary.

Definition 2.1. Let $\mathcal{T} \subset \mathbb{R}^N$ be a Lipschitz domain. Let $u \in V((-\Delta)^s_{\mathcal{T}}, \mathcal{T}) := \{u \in H^s(\mathcal{T}) : (-\Delta)^s_{\mathcal{T}} u \in L^2(\mathcal{T})$ in the sense of distributions$\}$. We say that u has a weak fractional normal derivative in $(H^{s-\frac{1}{2}}(\partial\mathcal{T}))'$ if there exists $g \in (H^{s-\frac{1}{2}}(\partial\mathcal{T}))'$ such that

$$\langle g, v \rangle_{(H^{s-\frac{1}{2}}(\partial\mathcal{T}))', H^{s-\frac{1}{2}}(\partial\mathcal{T})} = -\int\limits_{\mathcal{T}} (-\Delta)^s_{\mathcal{T}} u \, v \,\mathrm{d}\mathcal{L}_N \tag{2.2}$$

$$+ \frac{C_{N,s}}{2} \iint\limits_{\mathcal{T}\times\mathcal{T}} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x) \mathrm{d}\mathcal{L}_N(y)$$

for every $v \in H^s(\mathcal{T})$. In this case, g is uniquely determined and we call $C_s \mathcal{N}_{2-2s} u := g$ the weak fractional normal derivative of u, where

$$C_s := \frac{C_{1,s}}{2s(2s-1)} \int\limits_0^\infty \frac{|t-1|^{1-2s} - (t \vee 1)^{1-2s}}{t^{2-2s}} \,\mathrm{d}t.$$

We point out that, when $s \to 1^-$ in (2.2), we recover the usual Green formula for Lipschitz domains.

We now prove a "fractional" Green formula for (ε, δ) domains Q having as boundary ∂Q a closed set supporting a Borel measure satisfying (1.3). We suppose that Q can be approximated by a sequence $\{Q_n\}$ of domains such that for every $n \in \mathbb{N}$:

1) Q_n is bounded and Lipschitz;

2) $Q_n \subseteq Q_{n+1}$;

3) $Q = \bigcup\limits_{n=1}^{\infty} Q_n$.

We define the space

$$V((-\Delta)^s_Q, Q) := \{u \in H^s(Q) \,:\, (-\Delta)^s_Q u \in L^2(Q) \text{ in the sense of distributions}\},$$

which is a Banach space when equipped with the norm

$$\|u\|^2_{V((-\Delta)^s_Q, Q)} := \|u\|^2_{H^s(Q)} + \|(-\Delta)^s_Q u\|^2_{L^2(Q)}.$$

Theorem 2.2 (Fractional Green formula)**.** *There exists a bounded linear operator $\mathcal{N}_{2-2s}$ from $V((-\Delta)^s_Q, Q)$ to $(\tilde{B}^{2,2}_s(\partial Q))'$.*
The following generalized Green formula holds for all $u \in V((-\Delta)^s_Q, Q)$:

$$C_s \,\langle \mathcal{N}_{2-2s} u, v \rangle_{(\tilde{B}^{2,2}_s(\partial Q))', \tilde{B}^{2,2}_s(\partial Q)} = -\int\limits_Q (-\Delta)^s_Q u \, v \,\mathrm{d}\mathcal{L}_N + \frac{C_{N,s}}{2} (u, v)_s, \quad v \in H^s(Q). \tag{2.3}$$

Proof. For $u \in V((-\Delta)^s_Q, Q)$ and $v \in H^s(Q)$, we define

$$\langle l(u), v \rangle := -\int\limits_Q (-\Delta)^s_Q u \, v \,\mathrm{d}\mathcal{L}_N + \frac{C_{N,s}}{2} (u, v)_s.$$

From Cauchy-Schwarz Theorem and trace results we get

$$\begin{aligned} |\langle l(u), v \rangle| &\le \|(-\Delta)^s_Q u\|_{L^2(Q)} \|v\|_{L^2(Q)} + \frac{C_{N,s}}{2} \|u\|_{H^s(Q)} \|v\|_{H^s(Q)} \\ &\le c\, \|u\|_{V((-\Delta)^s_Q, Q)} \|v\|_{H^s(Q)} \le c\, \|u\|_{V((-\Delta)^s_Q, Q)} \|v\|_{\tilde{B}^{2,2}_s(\partial Q)}. \end{aligned}$$

This shows in particular that the operator is independent from the choice of v and it is an element of the dual space of $\tilde{B}^{2,2}_s(\partial Q)$.
We now consider the sequence of domains Q_n satisfying the properties 1)-2)-3) above. We recall that for Lipschitz domains, the trace space of $H^s(Q_n)$, for $\frac{1}{2} < s < 1$, is $H^{s-\frac{1}{2}}(\partial Q_n)$. On these Lipschitz domains, by (2.2) the following fractional Green formula holds:

$$C_s \left\langle \mathcal{N}_{2-2s} u, v \right\rangle_{(H^{s-\frac{1}{2}}(\partial Q_n))', H^{s-\frac{1}{2}}(\partial Q_n)} = -\int\limits_Q \chi_{Q_n} (-\Delta)^s_Q u\, v \,\mathrm{d}\mathcal{L}_N$$

$$+ \frac{C_{N,s}}{2} \iint\limits_{Q\times Q} \chi_{Q_n}(x)\chi_{Q_n}(y) \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y).$$

From the dominated convergence theorem, we have

$$\lim_{n\to\infty} C_s \left\langle \mathcal{N}_{2-2s} u, v \right\rangle_{(H^{s-\frac{1}{2}}(\partial Q_n))', H^{s-\frac{1}{2}}(\partial Q_n)}$$
$$= \lim_{n\to\infty} \left(-\int\limits_{Q_n} (-\Delta)^s_Q u\, v \,\mathrm{d}\mathcal{L}_N + \frac{C_{N,s}}{2} \iint\limits_{Q_n\times Q_n} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) \right)$$
$$= -\int\limits_Q (-\Delta)^s_Q u\, v \,\mathrm{d}\mathcal{L}_N + \frac{C_{N,s}}{2} (u,v)_s = \langle l(u), v \rangle$$

for every $u \in V((-\Delta)^s_Q, Q)$ and $v \in H^s(Q)$. Hence, we define the fractional normal derivative on Q as

$$\left\langle C_s \mathcal{N}_{2-2s} u, v \right\rangle_{(\tilde{B}^{2,2}_s(\partial Q))', \tilde{B}^{2,2}_s(\partial Q)} := -\int\limits_Q (-\Delta)^s_Q u\, v \,\mathrm{d}\mathcal{L}_N + \frac{C_{N,s}}{2} (u,v)_s.$$

□

Remark 2.3. We remark that, when $s \to 1^-$ in (2.3), we recover the Green formula stated in [32] for fractal domains.
Let $u \in V(-\Delta, Q) := \{u \in H^1(Q) \,:\, -\Delta u \in L^2(Q) \text{ in the sense of distributions}\}$ and $v \in H^1(Q)$. It holds that

$$\lim_{s\to 1^-} \int\limits_Q (-\Delta)^s_Q u\, v \,\mathrm{d}\mathcal{L}_N = \int\limits_Q \nabla u\, \nabla v \,\mathrm{d}\mathcal{L}_N.$$

As first step, we take $v = u$ and $u \in C^\infty(\overline{Q})$. In particular then $u \in C^\infty(\overline{Q}_n)$ for every n and $\mathcal{N}_{2-2s} u = 0$ on ∂Q_n pointwise. From Theorem 2.2 we have

$$\lim_{s\to 1^-} \int\limits_Q \chi_{Q_n} u\, (-\Delta)^s_Q u \,\mathrm{d}\mathcal{L}_N = \lim_{s\to 1^-} \frac{(1-s)C_{N,s}}{2(1-s)} \iint\limits_{Q_n\times Q_n} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y). \tag{2.4}$$

Since the limit in the right-hand side of (2.4) is equal to $\int_{Q_n} |\nabla u|^2 \,\mathrm{d}\mathcal{L}_N$, passing to the limit as $n \to +\infty$ we get

$$\lim_{n\to+\infty} \lim_{s\to1^-} \int_Q \chi_{Q_n} u \,(-\Delta)^s_Q u \,\mathrm{d}\mathcal{L}_N = \lim_{n\to+\infty} \int_{Q_n} |\nabla u|^2 \,\mathrm{d}\mathcal{L}_N = \int_Q |\nabla u|^2 \,\mathrm{d}\mathcal{L}_N.$$

By density arguments we get the claim.

3 Energy forms and semigroups

We denote by $L^2(Q, m)$ the Lebesgue space with respect to the measure

$$\mathrm{d}m = \mathrm{d}\mathcal{L}_N + \mathrm{d}\mu, \tag{3.1}$$

where μ satisfies (1.3) on ∂Q. We endow $L^2(Q, m)$ with the following norm:

$$\|u\|^2_{L^2(Q,m)} = \|u\|^2_{L^2(Q)} + \|u\|^2_{L^2(\partial Q,\mu)}.$$

Let $b \in C(\overline{Q})$ be a strictly positive continuous function on $\overline{Q}$. We define the following energy functional for every $u \in H^s(Q)$:

$$E_s[u] = \frac{C_{N,s}}{2} |u|^2_{H^s(Q)} + \int_{\partial Q} b|u|^2 \,\mathrm{d}\mu = \frac{C_{N,s}}{2} \iint_{Q\times Q} \frac{(u(x) - u(y))^2}{|x - y|^{N+2s}} \,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) + \int_{\partial Q} b|u|^2 \,\mathrm{d}\mu. \tag{3.2}$$

Proposition 3.1. *$E_s[u]$ is closed on $L^2(Q, m)$.*

Proof. We have to prove that for every sequence $\{u_k\} \subseteq H^s(Q)$ such that

$$E_s[u_k - u_j] + \|u_k - u_j\|_{L^2(Q,m)} \to 0 \quad \text{for } k, j \to +\infty. \tag{3.3}$$

there exists $u \in H^s(Q)$ such that

$$E_s[u_k - u] + \|u_k - u\|_{L^2(Q,m)} \to 0 \quad \text{for } k \to +\infty.$$

(3.3) states that $\{u_k\}$ is a Cauchy sequence in $L^2(Q, m)$ and, since $L^2(Q, m)$ is a Banach space, there exists $u \in L^2(Q, m)$ such that

$$\|u_k - u\|_{L^2(Q,m)} \xrightarrow[k\to+\infty]{} 0.$$

This immediately implies that

$$\int_{\partial Q} b|u_k - u|^2 \,\mathrm{d}\mu \xrightarrow[k\to+\infty]{} 0$$

since b is a continuous function. Moreover, since $|u_k - u_j|_{H^s(Q)} + \|u_k - u_j\|_{L^2(Q)}$ is equivalent to the $H^s(Q)$-norm of $u_k - u_j$, (3.3) implies that $\{u_k\}$ is a Cauchy sequence also in $H^s(Q)$. Since $H^s(Q)$ is a Banach space, then also $|u_k - u|^2_{H^s(Q)} \to 0$ when $k \to +\infty$. $\square$

We define the bilinear form associated to the energy form $E_s[u]$ as follows: for every $u, v \in H^s(Q)$

$$E_s(u,v) = \frac{C_{N,s}}{2}(u,v)_s + \int_{\partial Q} b\, u\, v\, \mathrm{d}\mu.$$

The following result follows from [24].

Theorem 3.2. *For every $u, v \in H^s(Q)$, $E_s(u,v)$ is a closed symmetric bilinear form on $L^2(Q,m)$. Then there exists a unique self-adjoint non-positive operator A_s on $L^2(Q,m)$ such that*

$$E_s(u,v) = (-A_s u, v)_{L^2(Q,m)} \quad \forall\, u \in D(A_s), \forall\, v \in H^s(Q), \tag{3.4}$$

where $D(A_s) \subset H^s(Q)$ is the domain of A_s and it is dense in $L^2(Q,m)$.

Moreover, since $E_s[u]$ is closed, by proceeding as in Section 3 in [11], we prove that A_s is the generator of a strongly continuous semigroup $\{T_s(t)\}_{t\geq 0}$ on $L^2(Q,m)$.

Proposition 3.3. *$E_s[u]$ is Markovian on $L^2(Q,m)$, hence the semigroup $T_s(t)$ generated by A_s is Markovian.*

Proof. Since $E_s[u]$ is closed, it is sufficient to prove (see [11]) that for every strictly positive $u \in H^s(Q)$, the function $v := u \wedge 1 \in H^s(Q)$ and the following holds

$$E_s[v] \leq E_s[u].$$

The result trivially follows for the boundary term in E_s; as to the other term, the result follows from Lemma 2.7 in [41]. $\square$

We now prove some properties of the semigroup generated by A_s, following [14]. For the sake of completeness, we recall some definitions.

Definition 3.4. Let X be a locally compact metric space and $\tilde{\mu}$ be a Radon measure on X. Let $\{T(t)\}_{t\geq 0}$ be a strongly continuous semigroup on $L^2(X,\tilde{\mu})$. The semigroup is *positive preserving* if for every non-negative $u \in L^2(X,\tilde{\mu})$

$$T(t)u \geq 0 \quad \forall\, t \geq 0.$$

The semigroup is *L^∞-contractive* if for every $t \geq 0$

$$\|T(t)u\|_{L^\infty(X,\tilde{\mu})} \leq \|u\|_{L^\infty(X,\tilde{\mu})} \quad \forall\, u \in L^2(X,\tilde{\mu}) \cap L^\infty(X,\tilde{\mu}).$$

Theorem 3.5. *The semigroup $\{T_s(t)\}_{t\geq 0}$ generated by A_s is submarkovian on $L^2(Q,m)$, i.e. it is positive preserving and L^∞-contractive.*

Proof. The L^∞-contractivity follows from Theorem 1.4.1 in [11]. In order to prove that it is positive preserving, we follow the proof of Proposition 2.14 in [14]. We prove that $E_s[|u|] \leq E_s[u]$ for every $u \in H^s(Q)$; this condition is trivially fulfilled for the boundary term.
By recalling that $u = u^+ - u^-$ and $|u| = u^+ + u^-$, where u^+ and u^- are respectively the positive and negative part of u, the above condition reads as follows:

$$E_s[u^+ + u^-] \leq E_s[u^+ - u^-] \tag{3.5}$$

From Lemma 2.5 in [41] it follows that, if $u \in H^s(Q)$, then u^+ and u^- also belong to $H^s(Q)$. This implies that $|u| \in H^s(Q)$. It holds that

$$(|u|,|u|)_s = \iint\limits_{Q\times Q} \frac{(|u|(x) - |u|(y))^2}{|x-y|^{N+2s}}\, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) = \iint\limits_{Q\times Q} \frac{(u^+(x) - u^+(y))^2}{|x-y|^{N+2s}}\, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y)$$
$$+ \iint\limits_{Q\times Q} \frac{(u^-(x) - u^-(y))^2}{|x-y|^{N+2s}}\, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) + 2(u^+,u^-)_s.$$

Analogously, by calculation it follows that

$$(u,u)_s = \iint\limits_{Q\times Q} \frac{(u^+(x) - u^+(y))^2}{|x-y|^{N+2s}}\, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) + \iint\limits_{Q\times Q} \frac{(u^-(x) - u^-(y))^2}{|x-y|^{N+2s}}\, \mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) - 2(u^+,u^-)_s.$$

Hence condition (3.5) can be written as follows

$$(u^+,u^-)_s \leq 0,$$

which is true by inspection. From Theorem 1.3.2 in [7], this implies that the semigroup is positive preserving. □

Theorem 3.6. *Let $\lambda_1 > 0$ be the first eigenvalue of $-A_s$. The semigroup $\{T_s(t)\}_{t\geq 0}$ is ultracontractive, i.e. $T_s(t)\colon L^2(Q,m) \to L^\infty(Q,m)$. Moreover, for every $1 \leq q \leq p \leq \infty$ there exists a positive constant C such that $\forall\, u \in L^q(Q,m)$ and for every $t > 0$*

$$\|T_s(t)u\|_{L^p(Q,m)} \leq C e^{-\lambda_1(\frac{1}{q}-\frac{1}{p})t} t^{-\frac{N}{2s}} \|u\|_{L^q(Q,m)}. \tag{3.6}$$

Proof. The proof follows from Theorem 2.16 in [14] with small suitable changes. □

4 The evolution problems

In order to study the problems $(\tilde{P})$ and $(\tilde{P}_n)$ stated in the Introduction, we first consider the following abstract Cauchy problem, for $T > 0$ fixed:

$$(P) \begin{cases} u'(t) = A_s u(t) + f(t) & \text{for } t \in (0,T], \\ u(0) = u_0, \end{cases} \tag{4.1}$$

where f and u_0 are given functions in suitable spaces and A_s is the operator associated to the energy form E_s. From semigroup theory we get the following existence and uniqueness result.

Theorem 4.1. *Let $\alpha \in (0,1)$, $f \in C^{0,\alpha}([0,T]; L^2(Q,m))$ and $u_0 \in \overline{D(A_s)}$. We define*

$$u(t) = T_s(t)\, u_0 + \int_0^t T_s(t-\tau)\, f(\tau)\, \mathrm{d}\tau, \tag{4.2}$$

where $T_s(t)$ is the semigroup generated by the operator A_s. Then u defined in (4.2) is the unique classical *solution of problem (P), i.e. a function u such that $u'(t) = A_s u(t) + f(t)$ for all $t \in (0,T]$, $u(0) = u_0$ and*

$$u \in C^1((0,T]; L^2(Q,m)) \cap C((0,T]; D(A_s)) \cap C([0,T]; L^2(Q,m)).$$

Moreover, the following estimate holds:

$$\|u\|_{C((0,T];L^2(Q,m))} \leq C(\|u_0\|_{L^2(Q,m)} + \|f\|_{C^{0,\alpha}([0,T];L^2(Q,m))}).$$

For the proof see Theorem 4.3.1 and Corollary 4.2.4 in [34]. We now give a strong interpretation of the abstract Cauchy problem (P).

Theorem 4.2. *Let u be the unique solution of problem (P). Then, for every $t \in (0,T]$, it holds that*

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) + (-\Delta)^s_Q u(t,x) = f(t,x) & \textit{for a.e. } x \in Q, \\ \frac{\partial u}{\partial t} + C_s \mathcal{N}_{2-2s} u + bu = f & \textit{in } (\tilde{B}^{2,2}_s(\partial Q))', \\ u(0,x) = u_0(x) & \textit{in } \overline{Q}. \end{cases} \tag{4.3}$$

Proof. For every fixed $t \in (0,T]$, we multiply the first equation of problem (P) by a test function $\varphi \in D(Q)$ and then we integrate on Q. Then from (3.4) we obtain

$$\int_Q \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}\mathcal{L}_N = \int_Q A_s u\,\varphi\,\mathrm{d}\mathcal{L}_N + \int_Q f\,\varphi\,\mathrm{d}\mathcal{L}_N = -E_s(u,\varphi) + \int_Q f\,\varphi\,\mathrm{d}\mathcal{L}_N.$$

Since φ has compact support in Q, after integrating by parts, we get

$$\frac{\partial u}{\partial t} + (-\Delta)^s_Q u = f \quad \text{in } (D(Q))'. \tag{4.4}$$

By density, equation (4.4) holds in $L^2(Q)$, so it holds for a.e. $x \in Q$. We remark that from this it follows that, for each fixed $t \in (0,T]$, $u \in V((-\Delta)^s_Q, Q)$. Hence, we can apply Green formula (2.3).

We now take the scalar product in $L^2(Q,m)$ between the first equation of problem (P) and $\varphi \in H^s(Q)$. Hence we get

$$\left(\frac{\partial u}{\partial t}, \varphi\right)_{L^2(Q,m)} = (A_s u, \varphi)_{L^2(Q,m)} + (f,\varphi)_{L^2(Q,m)}. \tag{4.5}$$

Again by using (3.4), we have that

$$\int_Q \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}\mathcal{L}_N + \int_{\partial Q} \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}\mu = -\frac{C_{N,s}}{2}\iint_{Q\times Q} \frac{(u(x)-u(y))(\varphi(x)-\varphi(y))}{|x-y|^{N+2s}}\,\mathrm{d}\mathcal{L}_N(x)\mathrm{d}\mathcal{L}_N(y) - \int_{\partial Q} b\,u\,\varphi\,\mathrm{d}\mu + \int_Q f\,\varphi\,\mathrm{d}\mathcal{L}_N + \int_{\partial Q} f\,\varphi\,\mathrm{d}\mu.$$

Using (2.3) and (4.4), we obtain for every $\varphi \in H^s(Q)$ and for each $t \in (0,T]$

$$\int_{\partial Q} \frac{\partial u}{\partial t}\,\varphi\,\mathrm{d}\mu = -\left\langle C_s \mathcal{N}_{2-2s} u, \varphi\right\rangle - \int_{\partial Q} b\,u\,\varphi\,\mathrm{d}\mu + \int_{\partial Q} f\,\varphi\,\mathrm{d}\mu. \tag{4.6}$$

Hence the boundary condition holds in $(\tilde{B}^{2,2}_s(\partial Q))'$. $\square$

We point out that, by introducing the Lebesgue space $L^2(Q,m_n)$ with respect to the measure

$$\mathrm{d}m_n = \chi_{Q_n}\mathrm{d}\mathcal{L}_N + \chi_{\partial Q_n}\mathrm{d}\mathcal{L}_{N-1} \tag{4.7}$$

and the energy functional $E^{(n)}_s[u]$, for $u \in H^s(Q)$, with the obvious changes, the previous results hold also for the approximating domains Q_n introduced in Section 2. In particular, since the Lipschitz boundary ∂Q_n is a $(N-1)$-set, we can define the fractional normal derivative $\mathcal{N}_{2-2s}u$ on ∂Q_n as an element of $(H^{s-\frac{1}{2}}(\partial Q_n))'$.

5 The fractal problem

We now consider a particular (ε,δ) domain Q and its approximating Lipschitz domains Q_n. We will study problems $(\tilde{P})$ and $(\tilde{P}_n)$ specialized to this case. A crucial problem

is to understand whether the solutions of these problems exist and in which sense they can be approximated. This is a step towards the numerical approximation, which will be object of a further research.
We consider the case of a "fractal cylinder" of Koch type. More precisely, Q denotes the open bounded set defined as the Cartesian product of the snowflake domain Ω and the unit interval; the "lateral surface" S is the product of the Koch snowflake F and $I = [0, 1]$, and the bases are the sets $\Omega \times \{0\}$ and $\Omega \times \{1\}$ (see Figure 1). For details on the fractal sets, see [31].

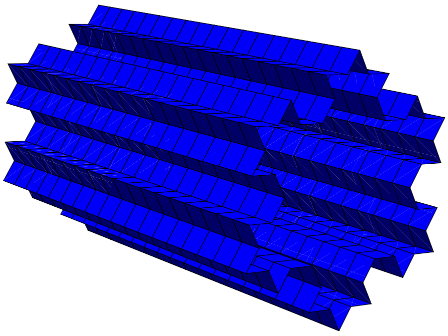

Figure 1: The fractal domain Q.

We introduce on S the measure

$$\mathrm{d}g = \mathrm{d}\mu \times \mathrm{d}\mathcal{L}_1, \tag{5.1}$$

where μ is the d_f-normalized Hausdorff measure on F (see [9, 10]), $d_f := \frac{\ln 4}{\ln 3}$ is the Hausdorff dimension of F and $\mathcal{L}_1$ is the one-dimensional Lebesgue measure on I. We remark that S is a $(d_f + 1)$-set, while the boundary $\partial Q = S \cup (\Omega \times \{0\}) \cup (\Omega \times \{1\})$ is neither a 2-set nor a $(d_f + 1)$-set; ∂Q is a closed set of $\mathbb{R}^3$. We define the measure $\tilde{\mu}$ supported on ∂Q as

$$\mathrm{d}\tilde{\mu} = \chi_S \mathrm{d}g + \chi_{\tilde{\Omega}} \mathrm{d}\mathcal{L}_2,$$

where $\tilde{\Omega} = (\Omega \times \{0\}) \cup (\Omega \times \{1\})$. The measure $\tilde{\mu}$ satisfies condition (1.3) with $d_1 = 2$ and $d_2 = d_f + 1$.
On this domain, we consider the following problem:

$$\begin{cases} \frac{\partial u}{\partial t} + (-\Delta)^s_Q u = f & \text{in } Q, \\ \frac{\partial u}{\partial t} + C_s \mathcal{N}_{2-2s} u + bu = f & \text{on } S, \\ u = 0 & \text{on } \tilde{\Omega}, \\ u(0, x) = u_0(x) & \text{in } \overline{Q}. \end{cases} \tag{5.2}$$

Since we are considering mixed boundary conditions, by proceeding as in [27] and following the patterns of Theorem 2.2, we can prove a fractional Green formula which in turn allows to prove that $\mathcal{N}_{2-2s}u \in (B^{2,2}_{\eta,0}(S))'$, where

$$B^{2,2}_{\eta,0}(S) = \{w \in L^2(S) \,:\, \exists\, v \in H^s(Q) \text{ s.t. } v = 0 \text{ on } \tilde{\Omega} \text{ and } \gamma_0 v = w \text{ on } S\}.$$

The fractal energy functional is defined as follows:

$$E_s[u] = \frac{C_{3,s}}{2} \iint\limits_{Q\times Q} \frac{(u(x)-u(y))^2}{|x-y|^{2s+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \int\limits_S b|u|^2\, \mathrm{d}g, \tag{5.3}$$

with domain

$$\mathcal{D}(E_s) := \{u \in H^s(\Omega) \,:\, u = 0 \text{ on } \tilde{\Omega}\}.$$

We define the measure

$$\mathrm{d}m = \mathrm{d}\mathcal{L}_3 + \mathrm{d}g. \tag{5.4}$$

The following result holds, by proceeding as in Section 3.

Proposition 5.1. *The energy form E_s with domain $\mathcal{D}(E_s)$ is closed in $L^2(Q, m)$.*

Following the patterns of Section 3, with the obvious changes, it can be proved that the solution of the abstract Cauchy problem associated with the generator $-A_s$ of the form $(E_s, \mathcal{D}(E_s))$ solves the following problem for every $t \in (0, T]$:

$$(\bar{P}) \begin{cases} \frac{\partial u}{\partial t}(t,x) + (-\Delta)^s_Q u(t,x) = f(t,x) & \text{for a.e. } x \in Q, \\ \frac{\partial u}{\partial t} + C_s \mathcal{N}_{2-2s} u + bu = f & \text{in } (B^{2,2}_{\eta,0}(S))', \\ u(t,x) = 0 & \text{in } H^{s-\frac{1}{2}}(\tilde{\Omega}), \\ u(0,x) = u_0(x) & \text{in } L^2(\overline{Q}, m), \end{cases} \tag{5.5}$$

where $\eta := s - 1 + \frac{d_f}{2} > 0$.

The fractal domain Q can be approximated by a sequence of invading Lipschitz domains $\{Q_n\}_{n\in\mathbb{N}}$; these sets are the so-called pre-fractal domains, and they satisfy the assumptions 1)-2)-3) of Section 2. These pre-fractal sets have as lateral surface $S_n = F_n \times I$, where F_n is the n-th approximation of the Koch snowflake F; since F_n is Lipschitz, we can define in a natural way the arc length ℓ on F_n. For details on the construction of Q_n, see [31].

We consider now the approximating pre-fractal energy functionals $E^{(n)}_s$ for every $n \in \mathbb{N}$:

$$E^{(n)}_s[u] = \frac{C_{3,s}}{2} \iint\limits_{Q\times Q} \chi_{Q_n}(x)\chi_{Q_n}(y) \frac{(u(x)-u(y))^2}{|x-y|^{2s+3}}\, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \delta_n \int\limits_{S_n} b|u|^2\, \mathrm{d}\sigma, \tag{5.6}$$

where $d\sigma = d\ell \times d\mathcal{L}_1$ is the measure on every affine face of S_n and δ_n is a fixed positive parameter, with domain

$$\mathcal{D}(E_s^{(n)}) := \{u \in H^s(Q) \, : \, u = 0 \text{ on } \tilde{\Omega}_n\},$$

where $\tilde{\Omega}_n = (\Omega_n \times \{0\}) \cup (\Omega_n \times \{1\})$.
By $E_s^{(n)}(u, v)$ we denote the bilinear form induced by the quadratic form $E_s^{(n)}[u]$ in the obvious way. Following the patterns of Section 3, we can prove that for every $n \in \mathbb{N}$ the energy functional $E_s^{(n)}$ enjoys the same properties of the functional E_s as in Proposition 5.1; in particular, if we denote by $L^2(Q, m_n)$ the Lebesgue space with respect to the measure

$$dm_n = \chi_{Q_n} d\mathcal{L}_3 + \chi_{S_n} \delta_n d\sigma, \tag{5.7}$$

we can prove that there exists a unique self-adjoint non-positive operator $A_s^{(n)}$ on $L^2(Q, m_n)$ having domain $D(A_s^{(n)}) \subset \mathcal{D}(E_s^{(n)})$ dense in $L^2(Q, m_n)$ such that

$$E_s^{(n)}(u, v) = (-A_s^{(n)} u, v)_{L^2(Q,m_n)} \quad \forall \, u \in D(A_s^{(n)}), \forall \, v \in \mathcal{D}(E_s^{(n)}). \tag{5.8}$$

Moreover, this operator is the generator of a strongly continuous semigroup $\{T_s^{(n)}(t)\}_{t\geq 0}$ on $L^2(Q, m_n)$, which as in the fractal case is submarkovian on $L^2(Q, m_n)$ and ultracontractive.
We now consider, for every $n \in \mathbb{N}$, the following abstract Cauchy problem:

$$(P_n) \begin{cases} u_n'(t) = A_s^{(n)} u_n(t) + f_n(t) & \text{for } t \in (0, T], \\ u_n(0) = u_0^{(n)}. \end{cases} \tag{5.9}$$

The following analogue of Theorem 4.1 holds in the pre-fractal case.

Theorem 5.2. *Let $\alpha \in (0, 1)$, $f_n \in C^{0,\alpha}([0, T]; L^2(Q, m_n))$ and $u_0^{(n)} \in \overline{D(A_s^{(n)})}$. We define*

$$u_n(t) = T_s^{(n)}(t)\, u_0^{(n)} + \int_0^t T_s^{(n)}(t - \tau)\, f_n(\tau)\, d\tau, \tag{5.10}$$

where $T_s^{(n)}(t)$ is the semigroup generated by the operator $A_s^{(n)}$. Then u_n defined in (5.10) is the unique classical solution of problem (P_n), i.e. $u_n'(t) = A_s^{(n)} u_n(t) + f_n(t)$ for all $t \in (0, T]$, $u_n(0) = u_0^{(n)}$ and

$$u_n \in C^1((0, T]; L^2(Q, m_n)) \cap C((0, T]; D(A_s^{(n)})) \cap C([0, T]; L^2(Q, m_n)).$$

Moreover, the following estimate holds:

$$\|u_n\|_{C((0,T];L^2(Q,m_n))} \leq C(\|u_0^{(n)}\|_{L^2(Q,m_n)} + \|f_n\|_{C^{0,\alpha}([0,T];L^2(Q,m_n))}), \tag{5.11}$$

where C is a constant independent of n.

We introduce the space

$$H^{s-\frac{1}{2}}_{0,0}(S_n) = \{u \in L^2(S_n) \, : \, \exists \, v \in H^s(\Omega) \text{ s.t. } v = 0 \text{ on } \tilde{\Omega} \text{ and } \gamma_0 v = u \text{ on } S_n\}.$$

By proceeding as in the proof of Theorem 4.2, we can prove the following result.

Theorem 5.3. *For every $n \in \mathbb{N}$, let u_n be the unique solution of problem (P_n). Then, for every $t \in (0, T]$, it holds that*

$$(\bar{P}_n) \begin{cases} \frac{\partial u_n}{\partial t}(t,x) + (-\Delta)^s_{Q_n} u_n(t,x) = f_n(t,x) & \text{for a.e. } x \in Q_n, \\ \delta_n \frac{\partial u_n}{\partial t} + C_s \mathcal{N}_{2-2s} u_n + \delta_n b u_n = \delta_n f_n & \text{in } (H^{s-\frac{1}{2}}_{0,0}(S_n))', \\ u_n(t,x) = 0 & \text{in } H^{s-\frac{1}{2}}(\tilde{\Omega}_n), \\ u_n(0,x) = u_0^{(n)}(x) & \text{in } L^2(Q) \cap L^2(Q, m_n). \end{cases} \tag{5.12}$$

6 M-convergence of energy functionals

We study the convergence of the solutions of problems $(\bar{P}_n)$ to the solution of problem $(\bar{P})$. Since $\{Q_n\}$ is a sequence of domains which converges to Q when $n \to +\infty$, the natural setting for studying the convergence is that of varying Hilbert spaces introduced in Section 1.2.

We set $H = L^2(\overline{Q}, m)$, where m is the measure defined in (5.4), and the sequence $\{H_n\}_{n \in \mathbb{N}}$ with $H_n = \{L^2(Q) \cap L^2(Q, m_n)\}$ where m_n is the measure defined in (5.7), with norms

$$\|u\|^2_H = \|u\|^2_{L^2(Q)} + \|u\|^2_{L^2(S)}, \quad \|u\|^2_{H_n} = \|u\|^2_{L^2(Q_n)} + \delta_n \|u\|^2_{L^2(S_n)}.$$

The following results states the convergence of the sequence H_n to H in the sense of Definition 1.5. A key role is played by the choice of the factor δ_n.

Proposition 6.1. *Let $\delta_n = \left(\frac{3}{4}\right)^n$. Then the sequence $\{H_n\}_{n \in \mathbb{N}}$ converges in the sense of Definition 1.5 to H.*

For the proof, we refer to Proposition 5.13 in [29], where C and Z_n in Definition 1.5 are respectively $C(\overline{Q})$ and the identity operator on $C(\overline{Q})$.

We now introduce the notion of *M-convergence*. The definition of M-convergence of quadratic energy forms is due to Mosco [37] for a fixed Hilbert space; it was then adapted to the case of varying Hilbert spaces by Kuwae and Shioya (see Definition 2.11 in [26]).

Definition 6.2. Let H_n be a sequence of Hilbert spaces converging to a Hilbert space H. A sequence of forms $\left\{E_s^{(n)}\right\}$ defined in H_n M-converges to a form E_s defined in H if the following conditions hold:

a) for every $\{v_n\} \in H_n$ weakly converging to $u \in H$ in $\mathcal{H}$

$$\varliminf_{n\to\infty} E_s^{(n)}[v_n] \geq E_s[u];$$

b) for every $u \in H$ there exists a sequence $\{w_n\}$, with $w_n \in H_n$ strongly converging to u in $\mathcal{H}$, such that

$$\varlimsup_{n\to\infty} E_s^{(n)}[w_n] \leq E_s[u].$$

We extend the functionals E_s and $E_s^{(n)}$ to H and H_n respectively. We define

$$E_s[u] := \begin{cases} \dfrac{C_{3,s}}{2} \displaystyle\iint_{Q\times Q} \frac{(u(x)-u(y))^2}{|x-y|^{2s+3}} \, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \int_S b|u|^2 \, \mathrm{d}g & \text{if } u \in \mathcal{D}(E_s), \\ +\infty & \text{if } u \in H \setminus \mathcal{D}(E_s), \end{cases} \tag{6.1}$$

and, for every $n \in \mathbb{N}$,

$$E_s^{(n)}[u] := \begin{cases} \dfrac{C_{3,s}}{2} \displaystyle\iint_{Q\times Q} \chi_{Q_n}(x)\chi_{Q_n}(y) \frac{(u(x)-u(y))^2}{|x-y|^{2s+3}} \, \mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) & \\ +\delta_n \displaystyle\int_{S_n} b|u|^2 \, \mathrm{d}\sigma & \text{if } u \in \mathcal{D}(E_s^{(n)}), \\ +\infty & \text{if } u \in H_n \setminus \mathcal{D}(E_s^{(n)}). \end{cases} \tag{6.2}$$

We now prove two preliminary lemmas, before the main result (Theorem 6.5).

Proposition 6.3. *If $\{v_n\}_{n\in\mathbb{N}}$ weakly converges to a vector u in $\mathcal{H}$, then $\{v_n\}_{n\in\mathbb{N}}$ weakly converges to u in $L^2(Q)$ and $\lim_{n\to\infty} \delta_n \int_{S_n} \varphi v_n \, \mathrm{d}\sigma = \int_S \varphi u \, \mathrm{d}g$ for every $\varphi \in C(\overline{Q})$.*

For the proof see Proposition 6.6 in [29].

Proposition 6.4. *Let $v_n \rightharpoonup u$ in $H^s(Q)$ and $b \in C(\overline{Q})$. Then*

$$\delta_n \int_{S_n} b|v_n|^2 \, \mathrm{d}\sigma \to \int_S b|u|^2 \, \mathrm{d}g.$$

Proof. We adapt the proof of Proposition 3.7 in [5] to our case. It holds that

$$\left|\delta_n \int\limits_{S_n} b\,|v_n|^2\,\mathrm{d}\sigma - \int\limits_{S} b\,|u|^2\,\mathrm{d}g\right| \le \left|\delta_n \int\limits_{S_n} b\,|v_n|^2\,\mathrm{d}\sigma - \delta_n \int\limits_{S_n} b\,|u|^2\,\mathrm{d}\sigma\right|$$

$$+ \left|\delta_n \int\limits_{S_n} b\,|u|^2\,\mathrm{d}\sigma - \int\limits_{S} b\,|u|^2\,\mathrm{d}g\right| =: A_n + B_n.$$

For the term A_n, we have the following estimate:

$$A_n \le C\,\delta_n \|b\|_{C(\overline{Q})}\,\|v_n - u\|_{L^2(S_n)}\left(\|v_n\|_{L^2(S_n)} + \|u\|_{L^2(S_n)}\right).$$

Since v_n weakly converges to u in $H^s(Q)$ by hypothesis, v_n is equibounded in $H^s(Q)$; hence v_n strongly converges to u in $H^l(Q)$ for every $0 < l < s$.
Since Q has the extension property, we now consider the extension of $v_n - u$ in $H^l(\mathbb{R}^3)$. From Theorem 3.6 in [6] (see also [5]), it follows that, if $w \in H^{\tilde{\beta}}(\mathbb{R}^3)$, for $\frac{1}{2} < \tilde{\beta} \le \frac{3}{2}$,

$$\|w\|^2_{L^2(S_n)} \le \frac{C_{\tilde{\beta}}}{\delta_n}\|w\|^2_{H^{\tilde{\beta}}(\mathbb{R}^3)}, \tag{6.3}$$

where $C_{\tilde{\beta}}$ is independent of n. Moreover, from Theorem 1 on page 103 in [22], it follows that, for $0 < \tilde{\beta} < 1$, there exists a linear extension operator $\mathcal{E}\mathrm{xt} : H^{\tilde{\beta}}(Q) \to H^{\tilde{\beta}}(\mathbb{R}^3)$ such that

$$\|\mathcal{E}\mathrm{xt}\, w\|^2_{H^{\tilde{\beta}}(\mathbb{R}^3)} \le \bar{C}_{\tilde{\beta}}\|w\|^2_{H^{\tilde{\beta}}(Q)}, \tag{6.4}$$

with $\bar{C}_{\tilde{\beta}}$ depending on $\tilde{\beta}$. Therefore we get

$$\delta_n\,\|v_n - u\|_{L^2(S_n)} \le C\,\|v_n - u\|_{H^l(\mathbb{R}^3)} \le C\,\|v_n - u\|_{H^l(Q)}\,,$$

hence $A_n \to 0$ when $n \to +\infty$.
We now focus on B_n. Since $u \in H^s(Q)$, from [22, page 213] there exists a sequence $\{w_m\} \in C(\overline{Q}) \cap H^s(Q)$ such that $\|w_m - u\|_{H^s(Q)} \to 0$ as $m \to +\infty$. We then get

$$B_n \le \left|\delta_n \int\limits_{S_n} b\,|u|^2\,\mathrm{d}\sigma - \delta_n \int\limits_{S_n} b\,|w_m|^2\,\mathrm{d}\sigma\right| + \left|\delta_n \int\limits_{S_n} b\,|w_m|^2\,\mathrm{d}\sigma - \int\limits_{S} b\,|w_m|^2\,\mathrm{d}g\right|$$

$$+ \left|\int\limits_{S} b\,|w_m|^2\,\mathrm{d}g - \int\limits_{S} b\,|u|^2\,\mathrm{d}g\right|.$$

We proceed as above and estimate the first and the third term in the right-hand side with $\|u - w_m\|_{H^s(Q)}$, hence for every $\varepsilon > 0$ there exists $m_\varepsilon \in \mathbb{N}$ such that these two terms are less than $c\,\varepsilon$. Since $b\,w_m$ is a continuous function, if we take $m > m_\varepsilon$, the second term in the right-hand side goes to zero for $n \to +\infty$ from Proposition 6.1. $\square$

We now prove the main Theorem.

Theorem 6.5. *Let $\delta_n = (3^{1-d_f})^n = \left(\frac{3}{4}\right)^n$. Let E_s and $E_s^{(n)}$ be defined as in (6.1) and (6.2) respectively. Then $E_s^{(n)}$ M-converges to the functional E_s.*

Proof. We have to prove conditions a) and b) in Definition 6.2.
Proof of condition a). Let $v_n \in H_n$ be a weakly converging sequence in $\mathcal{H}$ to $u \in H$. We point out that we can suppose that $v_n \in \mathcal{D}(E_s^{(n)})$ and

$$\varliminf_{n\to\infty} E_s^{(n)}[v_n] < \infty,$$

otherwise the thesis follows trivially. These hypotheses imply that there exists a constant independent of n such that

$$\frac{C_{3,s}}{2} \iint\limits_{Q\times Q} \chi_{Q_n}(x)\chi_{Q_n}(y)\frac{(v_n(x)-v_n(y))^2}{|x-y|^{2s+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) + \delta_n \int\limits_{S_n} b|v_n|^2\,\mathrm{d}\sigma \le C. \tag{6.5}$$

In particular $\|v_n\|_{H^s(Q_n)} < C$. From Theorem 1 page 103 in [22], for every $n \in \mathbb{N}$ there exists a bounded linear operator $\mathrm{Ext}\colon H^s(Q_n) \to H^s(\mathbb{R}^3)$ such that

$$\|\mathrm{Ext}\, v_n\|_{H^s(\mathbb{R}^3)} \le C_{\mathrm{Ext}}\, \|v_n\|_{H^s(Q_n)} \le C_{\mathrm{Ext}}\, C,$$

with C_{Ext} independent of n.
We define $\hat{v}_n = \mathrm{Ext}\, v_n|_Q$. Then $\hat{v}_n \in H^s(Q)$ and $\|\hat{v}_n\|_{H^s(Q)} \le C_{\mathrm{Ext}}\, C$. Therefore there exists a subsequence (which we still denote by $\hat{v}_n$) weakly converging to some $\hat{v}$ in $H^s(Q)$ and strongly converging in $L^2(Q)$. From Proposition 6.3, v_n weakly converges to u in $L^2(Q)$. We now prove that $\hat{v} = u$ in $L^2(Q)$, that is

$$\int\limits_Q (\hat{v}-u)\varphi\,\mathrm{d}\mathcal{L}_3 = 0$$

for every $\varphi \in L^2(Q)$.
We first note that

$$\begin{aligned}\int\limits_Q (\hat{v}-u)\varphi\,\mathrm{d}\mathcal{L}_3 &= \int\limits_Q (\hat{v}-\hat{v}_n+\hat{v}_n-u)\varphi\,\mathrm{d}\mathcal{L}_3\\ &= \int\limits_Q (\hat{v}-\hat{v}_n)\varphi\,\mathrm{d}\mathcal{L}_3 + \int\limits_{Q_n} (v_n-u)\varphi\,\mathrm{d}\mathcal{L}_3 + \int\limits_{Q\setminus Q_n} (\hat{v}_n-u)\varphi\,\mathrm{d}\mathcal{L}_3.\end{aligned} \tag{6.6}$$

We claim that each term on the right-hand side of (6.6) tends to zero as $n \to +\infty$. From the strong convergence of $\hat{v}_n$ to $\hat{v}$ in $L^2(Q)$ and the weak convergence of v_n to u

in $L^2(Q)$, we deduce our claim for the first two terms. As to the third, from Hölder inequality we deduce that

$$\int\limits_{Q\setminus Q_n} |(\hat{v}_n - u)\varphi| \,\mathrm{d}\mathcal{L}_2 \leq \|\varphi\|_{L^2(Q\setminus Q_n)}(\|\hat{v}_n\|_{L^2(Q)} + \|u\|_{L^2(Q)}) \xrightarrow[n\to+\infty]{} 0$$

since $|Q \setminus Q_n| \to 0$ as $n \to +\infty$ and $\hat{v}_n$ is equibounded. Then we have that $\hat{v}_n \rightharpoonup u$ in $H^s(Q)$.

We now prove that

$$\varliminf_{n\to\infty} \iint\limits_{Q\times Q} \chi_{Q_n}(x)\chi_{Q_n}(y)\frac{(v_n(x)-v_n(y))^2}{|x-y|^{2s+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) \geq \iint\limits_{Q\times Q} \frac{(u(x)-u(y))^2}{|x-y|^{2s+3}}\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y). \tag{6.7}$$

We prove a preliminary fact. We recall that $\hat{v}_n$ converges to u weakly in $H^s(Q)$ and strongly in $L^2(Q)$.

We now set

$$\tilde{v}_n(x,y) := \chi_{Q_n}(x)\chi_{Q_n}(y)\frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{2s+3}{2}}}.$$

Since $\hat{v}_n$ belongs to $H^s(Q)$ and is equibounded, $\tilde{v}_n$ belongs to $L^2(Q\times Q)$ and is equibounded. Hence there exists a subsequence (still denoted by $\tilde{v}_n$) which weakly converges to $\tilde{v}$ in $L^2(Q\times Q)$. We claim that

$$\tilde{v}(x,y) = \tilde{u}(x,y) := \frac{u(x)-u(y)}{|x-y|^{\frac{2s+3}{2}}} \quad \text{a.e.}, \tag{6.8}$$

where u is the weak limit of $\hat{v}_n$ in $H^s(Q)$. We have to prove that

$$\iint\limits_{Q\times Q} (\tilde{v}(x,y) - \tilde{u}(x,y))\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) = 0 \quad \forall\, \varphi \in L^2(Q\times Q). \tag{6.9}$$

We point out that we can suppose that $\varphi \in C(Q\times Q)$; the thesis will then follow by density. We add and subtract the following two terms on the left-hand side of (6.9):

$$\iint\limits_{Q\times Q} \tilde{v}_n(x,y)\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) \quad \text{and} \quad \iint\limits_{Q\times Q} \frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{2s+3}{2}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y).$$

Hence the following holds:

$$\iint\limits_{Q\times Q} (\tilde{v}(x,y) - \tilde{u}(x,y))\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) = \iint\limits_{Q\times Q} (\tilde{v} - \tilde{v}_n)\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)$$

$$+ \iint\limits_{Q\times Q} \frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{2s+3}{2}}}(\chi_{Q_n}(x)\chi_{Q_n}(y) - \chi_Q(x)\chi_Q(y))\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)$$

$$+ \iint\limits_{Q\times Q} \left(\frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{2s+3}{2}}} - \frac{u(x)-u(y)}{|x-y|^{\frac{2s+3}{2}}}\right)\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) =: I_1^{(n)} + I_2^{(n)} + I_3^{(n)}.$$

We study these three terms separately. As to $I_1^{(n)}$, since $\tilde{v}_n$ weakly converges to $\tilde{v}$ in $L^2(Q\times Q)$,

$$I_1^{(n)} \xrightarrow[n\to+\infty]{} 0.$$

As to $I_2^{(n)}$, we point out that $\chi_Q(x)\chi_Q(y) - \chi_{Q_n}(x)\chi_{Q_n}(y) = \chi_{(Q\times Q)\setminus(Q_n\times Q_n)}(x,y)$, since $Q_n \subset Q$. Hence, from Hölder inequality it follows that

$$I_2^{(n)} = \iint\limits_{(Q\times Q)\setminus(Q_n\times Q_n)} \frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{2s+3}{2}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) \leq \|\hat{v}_n\|_{H^s(Q)}\|\varphi\|_{L^2((Q\times Q)\setminus(Q_n\times Q_n))},$$

and the right-hand side tends to zero as $n\to+\infty$ since $\hat{v}_n$ is equibounded in $H^s(Q)$.
As to $I_3^{(n)}$, we can rewrite it in the following way:

$$\begin{aligned}
I_3^{(n)} &= \iint\limits_{Q\times Q}\left(\frac{\hat{v}_n(x)-\hat{v}_n(y)}{|x-y|^{\frac{2s+3}{2}}}\varphi(x,y) - \frac{u(x)-u(y)}{|x-y|^{\frac{2s+3}{2}}}\varphi(x,y)\right)\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)\\
&= \iint\limits_{Q\times Q}\frac{\hat{v}_n(x)-u(x)}{|x-y|^{\frac{2s+3}{2}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y) - \iint\limits_{Q\times Q}\frac{\hat{v}_n(y)-u(y)}{|x-y|^{\frac{2s+3}{2}}}\varphi(x,y)\,\mathrm{d}\mathcal{L}_3(x)\mathrm{d}\mathcal{L}_3(y)\\
&= \int\limits_Q(\hat{v}_n(x)-u(x))\phi_1(x)\,\mathrm{d}\mathcal{L}_3(x) - \int\limits_Q(\hat{v}_n(y)-u(y))\phi_2(y)\,\mathrm{d}\mathcal{L}_3(y),
\end{aligned}$$

where

$$\phi_1(x) := \int\limits_Q\frac{\varphi(x,y)}{|x-y|^{\frac{2s+3}{2}}}\,\mathrm{d}\mathcal{L}_3(y), \quad \phi_2(y) := \int\limits_Q\frac{\varphi(x,y)}{|x-y|^{\frac{2s+3}{2}}}\,\mathrm{d}\mathcal{L}_3(x).$$

We point out that both ϕ_1 and ϕ_2 belong to $L^2(Q)$. Hence, since $\hat{v}_n$ converges strongly to u in $L^2(Q)$, from Hölder inequality also $I_3^{(n)}$ tends to zero as $n\to+\infty$, thus proving (6.9).
We remark that $\hat{v}_n = v_n$ on Q_n. Hence we have that

$$\chi_{Q_n}(x)\chi_{Q_n}(y)\frac{v_n(x)-v_n(y)}{|x-y|^{\frac{2s+3}{2}}} \rightharpoonup \frac{u(x)-u(y)}{|x-y|^{\frac{2s+3}{2}}}$$

in $L^2(Q\times Q)$. From the lower semicontinuity of the norm, we get (6.7).
The thesis then follows from (6.7), Proposition 6.4 and liminf properties of the sum.
Proof of condition b). We prove that for every $u\in H$ we can construct a sequence $\{w_n\}_{n\in\mathbb{N}}$ strongly converging to u in $\mathcal{H}$ such that

$$E_s[u] \geq \varlimsup_{n\to\infty} E_s^{(n)}[w_n].$$

We suppose that $u\in\mathcal{D}(E_s)$, otherwise $E_s[u] = +\infty$ and the thesis follows trivially from Lemma 1.12.
We set $w_n := u|_{Q_n}$. We have to prove that w_n strongly converges to u in $\mathcal{H}$; we use the characterization given in Lemma 1.9, i.e.

$(w_n, v_n)_{H_n} \to (u, v)_H$ for every sequence $\{v_n\}$ weakly converging to a vector v in $\mathcal{H}$.

By recalling the definitions of scalar products in H_n and H, from straightforward calculations it follows that

$$|(w_n, v_n)_{H_n} - (u, v)_H| = \left| \int\limits_{Q_n} w_n v_n \, \mathrm{d}\mathcal{L}_3 + \delta_n \int\limits_{S_n} w_n v_n \, \mathrm{d}\sigma - \int\limits_{Q} uv \, \mathrm{d}\mathcal{L}_3 - \int\limits_{S} uv \, \mathrm{d}g \right|$$

$$= \left| (w_n - u, v_n)_{L^2(Q_n)} + \delta_n \int\limits_{S_n} (w_n - u) v_n \, \mathrm{d}\sigma + (u, v_n)_{H_n} - (u, v)_H \right|$$

$$= |(u, v_n)_{H_n} - (u, v)_H| \xrightarrow[n \to +\infty]{} 0,$$

as $w_n = u$ on $\overline{Q}_n$ and v_n weakly converges to v in $\mathcal{H}$.

We now prove condition b) of Definition 6.2 for w_n. We have that

$$\overline{\lim_{n\to\infty}} E_s^{(n)}[w_n] = \overline{\lim_{n\to\infty}} \left(\frac{C_{3,s}}{2} \iint\limits_{Q_n \times Q_n} \frac{(w_n(x) - w_n(y))^2}{|x-y|^{2s+3}} \, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y) + \delta_n \int\limits_{S_n} b|w_n|^2 \, \mathrm{d}\sigma \right)$$

$$= \overline{\lim_{n\to\infty}} \left(\frac{C_{3,s}}{2} \iint\limits_{Q_n \times Q_n} \frac{(u(x) - u(y))^2}{|x-y|^{2s+3}} \, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y) + \delta_n \int\limits_{S_n} b|u|^2 \, \mathrm{d}\sigma \right)$$

$$= \frac{C_{3,s}}{2} \iint\limits_{Q \times Q} \frac{(u(x) - u(y))^2}{|x-y|^{2s+3}} \, \mathrm{d}\mathcal{L}_3(x) \mathrm{d}\mathcal{L}_3(y) + \int\limits_{S} b|u|^2 \, \mathrm{d}g = E_s[u],$$

where the last equality follows from Proposition 6.4 and conditions 1)-2)-3) in Section 2. This implies condition b) of Definition 6.2. $\square$

The M-convergence of the energy functionals implies the convergence of the semigroups, as stated in the following result.

Theorem 6.6. *Let $E_s^{(n)}$ and E_s be the energy functionals defined in (6.2) and (6.1) respectively. The sequence of semigroups $\left\{T_s^{(n)}(t)\right\}_{n\in\mathbb{N}}$ associated with $E_s^{(n)}$ converges to the semigroup $T_s(t)$ associated with E_s in the sense of Definition 1.13 for every $t > 0$.*

For the proof see [25, Theorem 2.8] and [26, Theorem 2.4].

Remark 6.7. All these results can be easily extended to the more general case of fractal mixture cylinders in $\mathbb{R}^3$, see e.g. [28] with obvious changes.

7 Convergence of the solutions of the abstract Cauchy problems

We consider the abstract Cauchy problems (P) and (P_n) introduced in Section 3 and 5 respectively. We recall that, from Theorem 4.1 and Theorem 5.2, these problems admit a unique classical solution respectively. We are interested in the asymptotic behavior of the sequence $\{u_n\}$ as $n \to +\infty$.

Let m and m_n be the measures defined in (5.4) and (5.7) respectively. We denote by $\mathrm{d}t$ the one-dimensional Lebesgue measure on $[0,T]$. We observe that $L^2([0,T] \times Q, \mathrm{d}t \times \mathrm{d}m_n)$ is isomorphic to $L^2([0,T]; H_n)$ and $L^2([0,T] \times \overline{Q}, \mathrm{d}t \times \mathrm{d}m)$ is isomorphic to $L^2([0,T]; H)$. If we define

$$K_n = L^2([0,T]; H_n) \text{ for every } n \in \mathbb{N} \text{ and } K = L^2([0,T]; H),$$

K_n converges to K in the sense of Definition 1.5, where the set C is now $C([0,T] \times \overline{Q})$ and Z_n is the identity operator on C.

We denote by $\mathcal{K} = \{\cup_n K_n\} \cup K$. We define strong and weak convergence in $\mathcal{K}$ according to Definition 1.6 and 1.7 respectively. In the following we use either the characterization of strong convergence in $\mathcal{K}$ given in Lemma 1.9 or the one given in Lemma 1.10. For the sake of clarity, we recall them.

Proposition 7.1. *A sequence of vectors $\{u_n\}_{n\in\mathbb{N}}$ strongly converges to u in $\mathcal{K}$ if one of the following holds:*

$$\text{a)} \quad \begin{cases} \displaystyle\int_0^T \|u_n(t)\|^2_{H_n}\,\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_0^T \|u(t)\|^2_H\,\mathrm{d}t \\ \displaystyle\int_0^T (u_n(t), \varphi(t))_{H_n}\,\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_0^T (u(t), \varphi(t))_H\,\mathrm{d}t \end{cases} \tag{7.1}$$

for every $\varphi \in C([0,T] \times \overline{Q})$;

$$\text{b)} \quad \int_0^T (u_n(t), v_n(t))_{H_n}\,\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_0^T (u(t), v(t))_H\,\mathrm{d}t \tag{7.2}$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ strongly converging to v in $\mathcal{K}$.

Remark 7.2. We point out that, by proceeding as in Proposition 6.3, the weak convergence in $\mathcal{K}$ implies the weak convergence in $L^2([0,T] \times Q)$.

Theorem 7.3. *Let u and u_n be the classical solutions of problems (P) and (P_n) given by Theorems 4.1 and 5.2 respectively. Let $\delta_n = \left(\frac{3}{4}\right)^n$. If for every $t \in [0,T]$ $f_n(t) \to f(t)$ strongly in $\mathcal{H}$, $u_0^{(n)} \to u_0$ strongly in $\mathcal{H}$ and there exists a constant C such that*

$$\|u_0^{(n)}\|_{\overline{D(A_s^{(n)})}} + \|f_n\|_{C^\alpha([0,T];H_n)} < C \quad \textit{for every } n \in \mathbb{N}, \tag{7.3}$$

then:

i) $\{u_n(t)\}$ converges to $u(t)$ in $\mathcal{H}$ for every fixed $t \in [0,T]$;

ii) $\{u_n\}$ converges to u in $\mathcal{K}$.

Proof. Since $u_0^{(n)} \to u_0$ and $f_n(t) \to f(t)$ in $\mathcal{H}$ for every $t \in [0,T]$, from Theorem 6.6 we have that for every $t \in [0,T]$

$$T_s^{(n)}(t) f_n(t) \xrightarrow[n\to+\infty]{} T_s(t) f(t) \quad \text{and} \quad T_s^{(n)}(t)\, u_0^{(n)} \xrightarrow[n\to+\infty]{} T_s(t) u_0 \quad \text{in } \mathcal{H}. \tag{7.4}$$

In order to prove $i)$, we use the characterization of the strong convergence given in Lemma 1.9. More precisely, we prove that for every $t \in [0,T]$

$$(u_n, v_n)_{H_n} \to (u, v)_H$$

for every sequence $\{v_n\}_{n\in\mathbb{N}}$ weakly converging in $\mathcal{H}$ to $v \in H$.
From (5.10), we get

$$\begin{aligned}
(u_n, v_n)_{H_n} &= \int_{Q_n} \left(\int_0^t T_s^{(n)}(t-\tau) f_n(\tau, x)\, \mathrm{d}\tau \right) v_n(x)\, \mathrm{d}\mathcal{L}_3 \\
&+ \delta_n \int_{S_n} \left(\int_0^t T_s^{(n)}(t-\tau) f_n(\tau, x(\ell))\, \mathrm{d}\tau \right) v_n(x(\ell))\, \mathrm{d}\sigma + (T_s^{(n)}(t)\, u_0^{(n)}, v_n)_{H_n} \\
&= \int_0^t \left(T_s^{(n)}(t-\tau) f_n(\tau), v_n \right)_{H_n} \mathrm{d}\tau + (T_s^{(n)}(t)\, u_0^{(n)}, v_n)_{H_n}.
\end{aligned}$$

From (7.4) and the weak convergence of v_n to v, we deduce for every $t \in [0,T]$

$$(T_s^{(n)}(t)\, u_0^{(n)}, v_n)_{H_n} + \left(T_s^{(n)}(t-\tau) f_n(\tau), v_n\right)_{H_n} \xrightarrow[n\to+\infty]{} (T_s(t)\, u_0, v)_H + (T(t-\tau) f(\tau), v)_H\,.$$

We recall that, for every $n \in \mathbb{N}$, $T_s^{(n)}(t)$ is a contraction semigroup. Hence, from Lemma 1.8 and (7.3) there exists a constant C independent from n such that

$$\left| (T_s^{(n)}(t)\, u_0^{(n)}, v_n)_{H_n} + \left(T_s^{(n)}(t-\tau) f_n(\tau), v_n\right)_{H_n} \right| \le C.$$

The claim then follows from the dominated convergence Theorem.
We now prove *ii*). From Proposition 7.1, this amounts to prove

$$\|u_n\|_{K_n} \to \|u\|_K \tag{7.5}$$

and

$$(u_n, \phi)_{K_n} \to (u, \phi)_K \quad \forall\, \phi \in C([0,T] \times \overline{Q}). \tag{7.6}$$

We note that from (5.11) and (7.3), for every $t \in (0,T]$ it holds that

$$\|u_n(t)\|_{H_n} \leq C(\|u_0^{(n)}\|_{L^2(Q,m_n)} + \|f_n\|_{C^{0,\alpha}([0,T];L^2(Q,m_n))}) \leq C,$$

where C is independent from n. Thus the sequence $\{u_n\}$ is equibounded in $[0,T]$, and from i) we get

$$\|u_n(t)\|_{H_n} \to \|u(t)\|_H\,.$$

Hence, from the dominated convergence Theorem we have that (7.5) holds.
We come to (7.6). From i), it follows in particular that for every $t \in [0,T]$

$$(u_n(t), \phi(t))_{H_n} \to (u(t), \phi(t))_H \quad \forall\, \phi \in C([0,T] \times \overline{Q}).$$

Since

$$|(u_n(t), \phi(t))_{H_n}| \leq C\,\|\phi\|_{C([0,T]\times\overline{Q})}\,,$$

again from the dominated convergence Theorem we deduce

$$(u_n, \phi)_{K_n} \xrightarrow[n\to+\infty]{} (u, \phi)_K,$$

thus proving (7.6). □

Remark 7.4. We point out that the convergence of f_n to f in $\mathcal{H}$, together with the equiboundedness property (7.3), imply the convergence of f_n in $\mathcal{K}$.

Remark 7.5. Since we assumed $u_0^{(n)} \in \overline{D(A_s^{(n)})}$, the solution is only classical. Hence $\frac{\partial u_n}{\partial t}$ can blow up for $t = 0$ and one cannot have the weak convergence of $\{\frac{\partial u_n}{\partial t}\}$ to $\frac{\partial u}{\partial t}$ in $\mathcal{K}$ and the convergence of $\{A_s^{(n)} u_n\}$ to A_s in $\mathcal{K}$. If we restrict to $[\varepsilon, T]$, the solution u_n belongs to the space $C([\varepsilon,T]; D(A_s^{(n)})) \cap C^1([\varepsilon,T]; L^2(Q,m_n))$ and the following a priori estimate holds (see Theorem 4.3.1 in [34]):

$$\|u_n\|_{C^1([\varepsilon,T];L^2(Q,m_n))} + \|u_n\|_{C([\varepsilon,T];D(A_s^{(n)}))} \leq C\left(\|u_0^{(n)}\|_{\overline{D(A_s^{(n)})}} + \|f_n\|_{C^{0,\alpha}([\varepsilon,T];L^2(Q,m_n))}\right),$$

where C is a constant independent of n.

We now define for $\varepsilon > 0$

$$K_n^{(\varepsilon)} = L^2([\varepsilon,T]; H_n),\; K^{(\varepsilon)} = L^2([\varepsilon,T]; H) \text{ and } \mathcal{K}^{(\varepsilon)} = \{\cup_n K_n^{(\varepsilon)}\} \cup K^{(\varepsilon)},$$

and we endow these spaces with the obvious scalar products.

Theorem 7.6. *Let the assumptions of Theorem 7.3 hold. We have*

i) $\left\{\frac{\partial u_n(t,x)}{\partial t}\right\}$ *weakly converges to* $\frac{\partial u(t,x)}{\partial t}$ *in* $\mathcal{K}^{(\varepsilon)}$*;*

ii) $\left\{A_s^{(n)} u_n\right\}$ *weakly converges to* $A_s u$ *in* $\mathcal{K}^{(\varepsilon)}$*.*

Proof. We first prove i). From Theorem 5.2 and (7.3) we have for every $n \in \mathbb{N}$

$$\sup_{t\in[\varepsilon,T]} \left\| \frac{\partial u_n}{\partial t} \right\|_{H_n} \leq C.$$

This implies in particular that $\frac{\partial u_n}{\partial t} \in L^2([\varepsilon, T]; H_n)$ and from Remark 7.5 there exists a constant C independent from n such that

$$\left\| \frac{\partial u_n}{\partial t} \right\|_{L^2([\varepsilon,T];H_n)} \leq C.$$

From Lemma 1.11 we deduce that there exists a subsequence (still denoted by $\frac{\partial u_n}{\partial t}$) weakly converging in $\mathcal{K}^{(\varepsilon)}$ to a function $v \in K^{(\varepsilon)}$. We have to show that $v = \frac{\partial u}{\partial t}$.
From the definition of weak convergence in $\mathcal{K}$ (which can be trivially extended to $\mathcal{K}^{(\varepsilon)}$) it follows that

$$\left(\frac{\partial u_n}{\partial t}, w_n \right)_{K_n} \xrightarrow[n\to+\infty]{} (v, w)_K$$

for every sequence $\{w_n\} \in K_n$ such that $w_n \to w$ in $\mathcal{K}$.
We take $w_n = \varphi$, where φ is an arbitrary function of $C^1([\varepsilon, T]; C^(\overline{Q}))$. We have

$$\lim_{n\to+\infty} \int_Q \int_\varepsilon^T \frac{\partial u_n(t,x)}{\partial t} \varphi(t,x)\, \mathrm{d}m_n \mathrm{d}t = \int_Q \int_\varepsilon^T v(t,x)\varphi(t,x)\, \mathrm{d}m \mathrm{d}t.$$

Proceeding as in Theorem 5.3 in [30], we integrate by parts in time and we get

$$\int_Q \int_\varepsilon^T \frac{\partial u_n(t,x)}{\partial t} \varphi(t,x)\, \mathrm{d}t \mathrm{d}m_n = \int_Q \left(u_n(T,x)\varphi(T,x) - u_n(\varepsilon,x)\varphi(\varepsilon,x)\right)\, \mathrm{d}m_n$$
$$- \int_Q \int_\varepsilon^T u_n(t,x) \frac{\partial \varphi(t,x)}{\partial t}\, \mathrm{d}t \mathrm{d}m_n. \tag{7.7}$$

Passing to the limit in the first term on the right-hand side of (7.7) as $n \to +\infty$, from condition i) in Theorem 7.3 we get

$$\int_Q (u_n(T,x)\varphi(T,x) - u_n(\varepsilon,x)\varphi(\varepsilon,x))\,\mathrm{d}m_n \xrightarrow[n\to+\infty]{} \int_Q (u(T,x)\varphi(T,x) - u(\varepsilon,x)\varphi(\varepsilon,x))\,\mathrm{d}m.$$

It remains to study

$$\lim_{n\to+\infty} \int_\varepsilon^T \int_Q u_n(t,x)\frac{\partial\varphi(t,x)}{\partial t}\,\mathrm{d}m_n\mathrm{d}t. \tag{7.8}$$

We observe that

$$\int_\varepsilon^T \int_Q u_n(t,x)\frac{\partial\varphi(t,x)}{\partial t}\,\mathrm{d}m_n\mathrm{d}t = \left(u_n(t), \frac{\partial\varphi(t)}{\partial t}\right)_{K_n^{(\varepsilon)}}.$$

From $ii)$ in Theorem 7.3 we have

$$\left(u_n(t), \frac{\partial\varphi(t)}{\partial t}\right)_{K_n^{(\varepsilon)}} \xrightarrow[n\to+\infty]{} \left(u(t), \frac{\partial\varphi(t)}{\partial t}\right)_{K^{(\varepsilon)}}.$$

Hence, passing to the limit as $n\to+\infty$ in (7.7) we have

$$\begin{aligned}\int_Q \int_\varepsilon^T v(t,x)\varphi(t,x)\,\mathrm{d}t\mathrm{d}m = &\int_Q (u(T,x)\varphi(T,x) - u(\varepsilon,x)\varphi(\varepsilon,x))\,\mathrm{d}m\\ &- \int_Q \int_\varepsilon^T u(t,x)\frac{\partial\varphi(t,x)}{\partial t}\,\mathrm{d}t\mathrm{d}m.\end{aligned}$$

Therefore $v = \frac{\partial u}{\partial t}$, i.e.

$$\frac{\partial u_n}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \quad \text{in } \mathcal{K}^{(\varepsilon)}. \tag{7.9}$$

The proof of $ii)$ can be done as in Theorem 5.3 of [30] taking into account Remark 7.4 with the obvious modifications. It holds

$$\lim_{n\to+\infty} \left(A_s^{(n)}u_n, u_n\right)_{K_n^{(\varepsilon)}} = (A_s u, u)_{K^{(\varepsilon)}},$$

hence

$$\lim_{n\to+\infty} \int_\varepsilon^T E_s^{(n)}[u_n(t)]\,\mathrm{d}t = \int_\varepsilon^T E_s[u(t)]\,\mathrm{d}t. \tag{7.10}$$

Hence the thesis follows. □

Remark 7.7. From $i)$ and $ii)$ of Theorem 7.6 and Remark 7.2 we have:

a) $\left\{\frac{\partial u_n}{\partial t}\right\}$ weakly converges to $\frac{\partial u}{\partial t}$ in $L^2([\varepsilon,T]\times Q)$;

b) $\left\{A_s^{(n)}u_n\right\}$ weakly converges to $A_s u$ in $L^2([\varepsilon,T]\times Q)$.

We now focus on the fractional normal derivative. We prove this preliminary result.

Proposition 7.8. *Let $\{w_n\}$, with $w_n\in K_n^{(\varepsilon)}$, be weakly convergent in $\mathcal{K}^{(\varepsilon)}$ to $w\in K^{(\varepsilon)}$. Then for every $\varphi\in L^2([\varepsilon,T];H^s(Q))$*

$$\int_\varepsilon^T (w_n,\varphi)_{L^2(S_n,\delta_n\sigma)}\,\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_\varepsilon^T (w,\varphi)_{L^2(S)}\,\mathrm{d}t.$$

Proof. We point out that, from Remark 7.2, for every $\varphi\in L^2([\varepsilon,T];H^s(Q))$ we have that

$$\int_\varepsilon^T (w_n,\varphi)_{L^2(Q)}\,\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_\varepsilon^T (w,\varphi)_{L^2(Q)}\,\mathrm{d}t.$$

From this it follows that

$$\int_\varepsilon^T (w_n,\varphi)_{L^2(Q_n)}\,\mathrm{d}t = \int_\varepsilon^T (w_n,\varphi)_{L^2(Q)}\,\mathrm{d}t - \int_\varepsilon^T (w_n,\varphi)_{L^2(Q\setminus Q_n)}\,\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_\varepsilon^T (w,\varphi)_{L^2(Q)}\,\mathrm{d}t, \tag{7.11}$$

since $\{w_n\}$ is equibounded. Moreover, by hypothesis we have that

$$\int_\varepsilon^T \left((w_n,\varphi)_{L^2(Q_n)}+(w_n,\varphi)_{L^2(S_n,\delta_n\sigma)}\right)\,\mathrm{d}t \xrightarrow[n\to+\infty]{} \int_\varepsilon^T \left((w,\varphi)_{L^2(Q)}+(w,\varphi)_{L^2(S)}\right)\,\mathrm{d}t. \tag{7.12}$$

The thesis then follows from (7.11) and (7.12). □

Theorem 7.9. *Under the assumptions of Theorem 7.3*

$$\int_\varepsilon^T \langle\mathcal{N}_{2-2s}u_n,\varphi\rangle_{(H_{0,0}^{s-\frac{1}{2}}(S_n))',H_{0,0}^{s-\frac{1}{2}}(S_n)} \xrightarrow[n\to+\infty]{} \int_\varepsilon^T \langle\mathcal{N}_{2-2s}u,\varphi\rangle_{(B^{2,2}_{\eta,0}(S))',B^{2,2}_{\eta,0}(S)} \tag{7.13}$$

for every $\varphi\in L^2([\varepsilon,T];H^s(Q))$, where $\eta=s-1+\frac{d_f}{2}$.

Proof. We take the scalar product in $L^2(S_n,\delta_n\sigma)$ between the second equation in problem $(\bar{P}_n)$ and $\varphi\in L^2([\varepsilon,T];H^s(Q))$ and then we integrate for $t\in[\varepsilon,T]$:

$$\int_\varepsilon^T\int_{S_n}\delta_n\frac{\partial u_n}{\partial t}\varphi\,\mathrm{d}\sigma\mathrm{d}t+\int_\varepsilon^T\langle C_s\mathcal{N}_{2-2s}u_n,\varphi\rangle_{(H_{0,0}^{s-\frac{1}{2}}(S_n))',H_{0,0}^{s-\frac{1}{2}}(S_n)}+\int_\varepsilon^T\int_{S_n}\delta_n b u_n\varphi\,\mathrm{d}\sigma\mathrm{d}t=\int_\varepsilon^T\int_{S_n}\delta_n f_n\varphi\,\mathrm{d}\sigma\mathrm{d}t. \tag{7.14}$$

Since $f_n \to f$ strongly in $\mathcal{H}$, from Remarks 7.4 and 7.2 in particular it follows that $f_n \to f$ in $L^2([\varepsilon, T] \times Q)$. Hence, from Proposition 7.8 the following holds for every $\varphi \in L^2([\varepsilon, T]; H^s(Q))$:

$$\lim_{n\to+\infty} \int_\varepsilon^T \langle C_s \mathcal{N}_{2-2s} u_n, \varphi \rangle_{(H_{0,0}^{s-\frac{1}{2}}(S_n))', H_{0,0}^{s-\frac{1}{2}}(S_n)}$$

$$= \lim_{n\to+\infty} \left(-\int_\varepsilon^T \int_{S_n} \delta_n \frac{\partial u_n}{\partial t} \varphi \, \mathrm{d}\sigma \mathrm{d}t - \int_\varepsilon^T \int_{S_n} \delta_n b u_n \varphi \, \mathrm{d}\sigma \mathrm{d}t + \int_\varepsilon^T \int_{S_n} \delta_n f_n \varphi \, \mathrm{d}\sigma \mathrm{d}t \right)$$

$$= -\int_\varepsilon^T \int_S \frac{\partial u}{\partial t} \varphi \, \mathrm{d}g \mathrm{d}t - \int_\varepsilon^T \int_S b u \varphi \, \mathrm{d}g \mathrm{d}t + \int_\varepsilon^T \int_S f \varphi \, \mathrm{d}g \mathrm{d}t = \int_\varepsilon^T \langle C_s \mathcal{N}_{2-2s} u, \varphi \rangle_{(B^{2,2}_{\eta,0}(S))', B^{2,2}_{\eta,0}(S)},$$

since u solves problem $(\bar{P})$.

$\square$

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References

[1] S. Abe, S. Thurner, *Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion*, Physica A, 356 (2005), 403–407.

[2] D. R. Adams, L. I. Hedberg, *Function Spaces and Potential Theory*, Springer-Verlag, Berlin, 1996.

[3] K. Bogdan, K. Burdzy, Z.-Q. Chen, *Censored stable processes*, Probab. Theory Related Fields, 127 (2003), 89–152.

[4] Z.-Q. Chen, T. Kumagai, *Heat kernel estimates for stable-like processes on d-sets*, Stoch. Process. Appl., 108 (2003), 27–62.

[5] S. Creo, M. R. Lancia, A. Vélez-Santiago, P. Vernole, *Approximation of a nonlinear fractal energy functional on varying Hilbert spaces*, Commun. Pure Appl. Anal., 17 (2018), 647–669.

[6] S. Creo, V. Regis Durante, *Convergence and density results for parabolic quasi-linear Venttsel' problems in fractal domains*, Discrete Contin. Dyn. Syst. Series S, 12 (2019), 65–90.

[7] E. B. Davies, *Heat Kernels and Spectral Theory*, Cambridge University Press, Cambridge, 1989.

[8] A. A. Dubkov, B. Spagnolo, V. V. Uchaikin, *Lèvy flight superdiffusion: An introduction*, Int. J. Bifurc. Chaos, 18 (2008), 2649–2672.

[9] K. Falconer, *The Geometry of Fractal Sets*, Cambridge University Press, Cambridge, 1986.

[10] U. R. Freiberg, M. R. Lancia, *Energy form on a closed fractal curve*, Z. Anal. Anwendungen, 23 (2004), 115–137.

[11] M. Fukushima, Y. Oshima, M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*, Walter de Gruyter and Co., Berlin, 1994.

[12] C. G. Gal, M. Warma, *Elliptic and parabolic equations with fractional diffusion and dynamic boundary conditions*, Evol. Equat. Control Theory, 5 (2016), 61–103.

[13] C. G. Gal, M. Warma, *Reaction-diffusion equations with fractional diffusion on non-smooth domains with various boundary conditions*, Discrete Contin. Dyn. Syst., 36 (2016), 1279–1319.

[14] C. G. Gal, M. Warma, *Nonlocal transmission problems with fractional diffusion and boundary conditions on non-smooth interfaces*, Comm. Partial Differential Equations, 42 (2017), 579–625.

[15] R. Gorenflo, F. Mainardi, A. Vivoli, *Continuous-time random walk and parametric subordination in fractional diffusion*, Chaos, Solitons, Fractals, 34 (2007), 87–103.

[16] Q. Y. Guan, *Integration by parts formula for regional fractional Laplacian.* Commun. Math. Phys., 266 (2006), 289–329.

[17] Q. Y. Guan, Z. M. Ma, *Boundary problems for fractional Laplacians*, Stoch. Dyn., 5 (2005), 385–424.

[18] Q. Y. Guan, Z. M. Ma, *Reflected symmetric α-stable processes and regional fractional Laplacian*, Probab. Theory Related Fields, 134 (2006), 649–694.

[19] M. Jara, *Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps*, Comm. Pure Appl. Math., 62 (2009), 198–214.

[20] P. W. Jones, *Quasiconformal mapping and extendability of functions in Sobolev spaces*, Acta Math., 147 (1981), 71–88.

[21] A. Jonsson, *Besov spaces on closed subsets of* $\mathbb{R}^n$, Trans. Amer. Math. Soc., 341 (1994), 355–370.

[22] A. Jonsson, H. Wallin, *Function Spaces on Subsets of* $\mathbb{R}^n$, Part 1, Math. Reports, vol.2, Harwood Acad. Publ., London, 1984.

[23] A. Jonsson, H. Wallin, *The dual of Besov spaces on fractals*, Studia Math., 112 (1995), 285–300.

[24] T. Kato, *Perturbation Theory for Linear Operators*, II edit., Springer-Verlag, Berlin-New York, 1976.

[25] A. V. Kolesnikov, *Convergence of Dirichlet forms with changing speed measures on* $\mathbb{R}^d$, Forum Math., 17 (2005), 225–259.

[26] K. Kuwae, T. Shioya, *Convergence of spectral structures: a functional analytic theory and its applications to spectral geometry,* Comm. Anal. Geom., 11 (2003), 599–673.

[27] M. R. Lancia, *A transmission problem with a fractal interface*, Z. Anal. Anwendungen, 21 (2002), 113–133.

[28] M. R. Lancia, V. Regis Durante, P. Vernole, *Density results for energy spaces on some fractafolds*, Z. Anal. Anwendungen, 34 (2015), 357–372.

[29] M. R. Lancia, V. Regis Durante, P. Vernole, *Asymptotics for Venttsel' problems for operators in non divergence form in irregular domains*, Discrete Contin. Dyn. Syst. Ser. S, 9 (2016), 1493–1520.

[30] M. R. Lancia, P. Vernole, *Convergence results for parabolic transmission problems across highly conductive layers with small capacity*, Adv. Math. Sci. Appl., 16 (2006), 411–445.

[31] M. R. Lancia, P. Vernole, *Irregular heat flow problems*, SIAM J. Math. Anal., 42 (2010), 1539–1567.

[32] M. R. Lancia, P. Vernole, *Semilinear fractal problems: approximation and regularity results*, Nonlinear Anal., 80 (2013), 216–232.

[33] M. R. Lancia, P. Vernole, *Venttsel' problems in fractal domains*, J. Evol. Equ., 14 (2014), 681–712.

[34] A. Lunardi, *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Progress in Nonlinear Differential Equations and their Applications, 16, Birkhäuser Verlag, Basel, 1995.

[35] B. B. Mandelbrot, J. W. Van Ness, *Fractional Brownian motions, fractional noises and applications*, SIAM Rev., 10 (1969), 422–437.

[36] A. Mellet, S. Mischler, C. Mouhot, *Fractional diffusion limit for collisional kinetic equations*, Arch. Ration. Mech. Anal., 199 (2011), 493–525.

[37] U. Mosco, *Convergence of convex sets and solutions of variational inequalities,* Advances in Math., 3 (1969), 510–585.

[38] W. R. Schneider, *Grey noise*, in: S. Albeverio, G. Casati, U. Cattaneo, D. Merlini, R. Moresi, eds. Stochastic Processes, Physics and Geometry, Teaneck, NJ, USA, World Scientific, 1990, 676–681.

[39] E. Valdinoci, *From the long jump random walk to the fractional Laplacian*, Bol. Soc. Esp. Mat. Apl. SeMA, 49 (2009), 33–44.

[40] L. Vlahos, H. Isliker, Y. Kominis, K. Hizonidis, *Normal and Anomalous Diffusion: A Tutorial*, in Order and Chaos, Vol. 10 (ed. T. Bountis), Patras University Press, 2008.

[41] M. Warma, *The fractional relative capacity and the fractional Laplacian with Neumann and Robin boundary conditions on open sets*, Potential Anal., 42 (2015), 499–547.